

Congruence Closure with Free Variables

**Haniel
Barbosa**¹

Pascal
Fontaine¹

Andrew
Reynolds²

¹University of Lorraine, CNRS, Inria, LORIA, Nancy, France

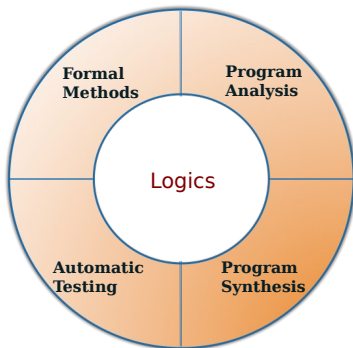
²University of Iowa, Iowa City, U.S.A.

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SMT 2017

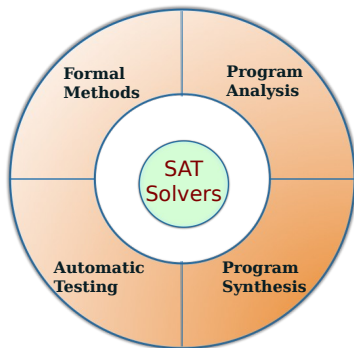
2017-07-22, Heidelberg, Germany

SMT solvers are successfully used in a variety of applications, including many verification tools



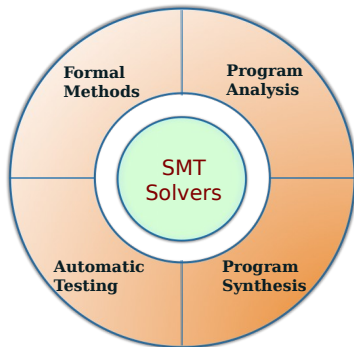
Picture credit: Vijay Ganesh

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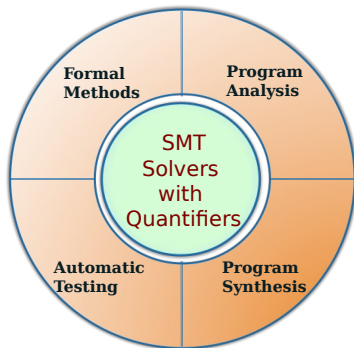
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Quantifiers in SMT solvers

Quantifiers primarily handled with heuristic instantiation

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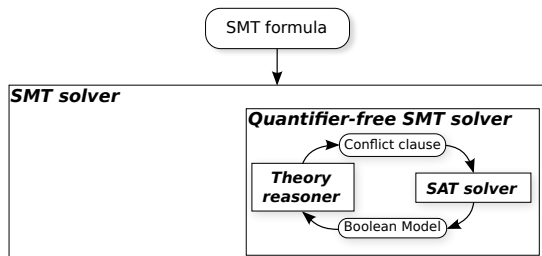
Fast semantically guided instantiation techniques

- ~~Too many instances swamp solver~~ **Fewer, necessary instances**
- ~~Butterfly effect~~ **Reduce dependency on heuristics**

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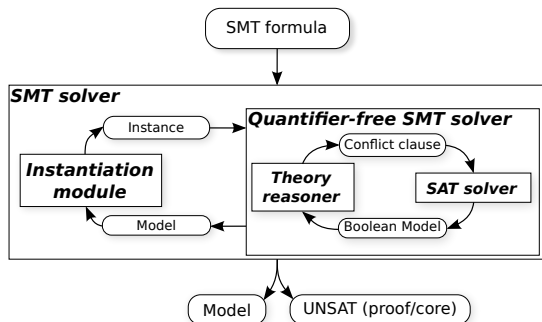
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- Derive instantiations that refute ground model**

Problem statement



- ▷ Quantifier-free solver enumerates models $E \cup Q$
 - ▶ E is a conjunctive set of ground literals
 - ▶ Q is a conjunctive set of quantified clauses

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 - ▶ E is a conjunctive set of ground literals
 - ▶ Q is a conjunctive set of quantified clauses
- ▷ Instantiation module generates instances from Q and adds them to E

Heuristic instantiation

Pattern-matching of terms from Q into terms of E

No consistency check of $E \cup Q$

- ⊖ Fast, but too many instances

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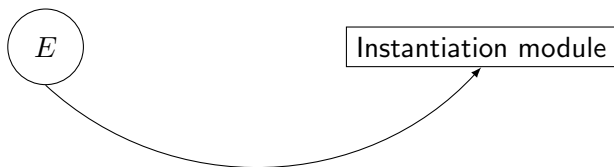
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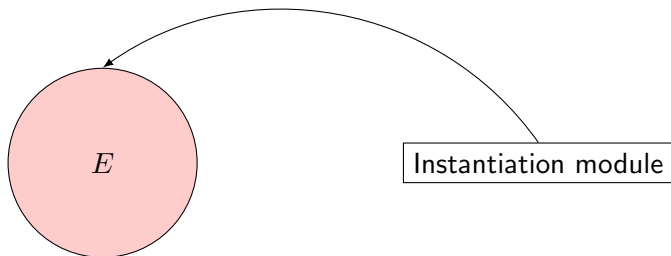


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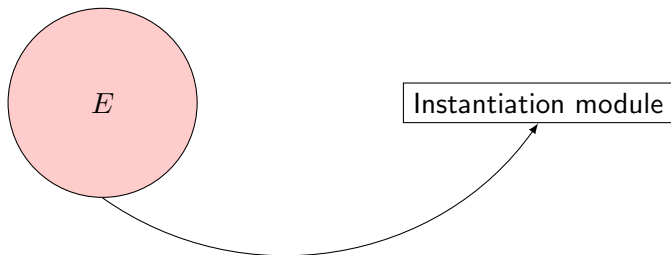


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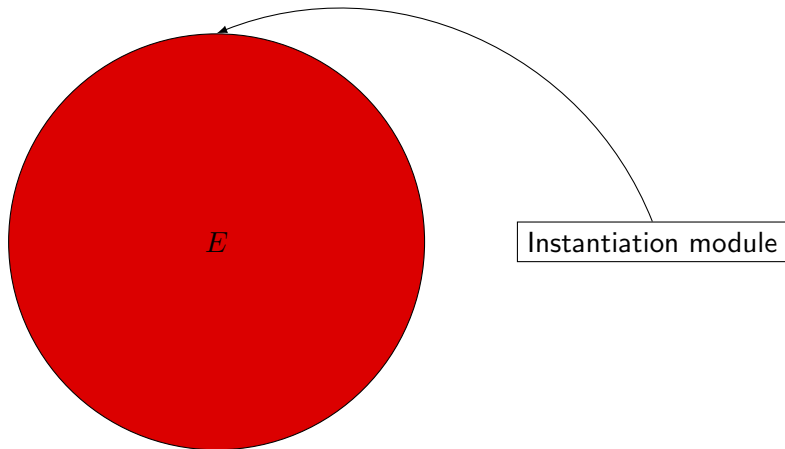


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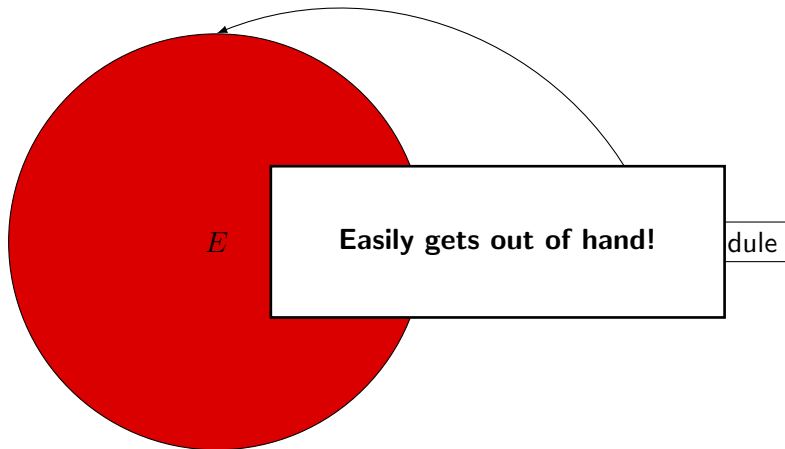


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Check consistency of $E \cup Q$

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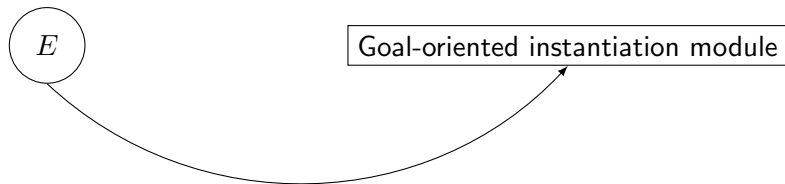


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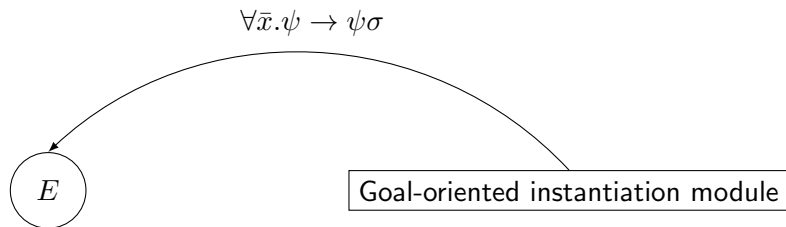
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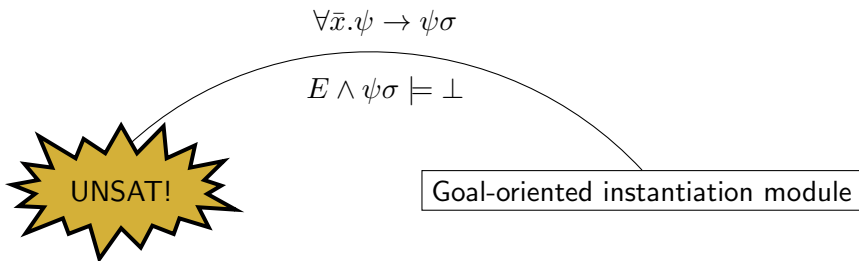
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Conflict-based instantiation

[RTM14]

- ▷ Given a model $E \cup Q$, for some $\forall \bar{x}. \psi \in Q$ find σ s.t. $E \wedge \psi\sigma \models \perp$
- ▷ Add instance $\forall \bar{x}. \psi \rightarrow \psi\sigma$ to quantifier-free solver

Finding conflicting instances requires deriving σ s.t.

$$E \models \neg\psi\sigma$$

- ⊕ Goal-oriented
- ⊕ Efficient
- ⊖ Ad-hoc
- ⊖ Incomplete

Let's look deeper into the problem

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▷ Variant of classic (non-simultaneous) rigid E -unification

▷ NP-complete

NP: Solutions can be restricted to ground terms in $E \cup L$

NP-hard: reduction of 3-SAT

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CCFV is a sound, complete and terminating calculus for solving E -ground (dis)unification

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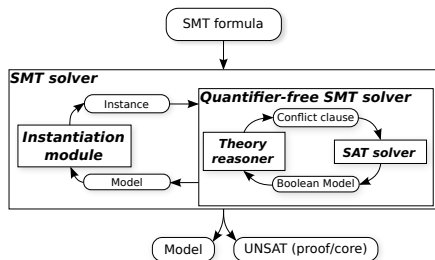
- ⊕ Goal-oriented
- ⊕ **(More) Efficient**
- ⊖ ~~Ad-hoc~~ **Versatile framework, recasting many instantiation techniques as a CCFV problem**
- ⊖ ~~Incomplete~~ **Finds all conflicting instances of a quantified formula**

Existing techniques as special cases

- ▷ Conflict-based instantiation [RTM14]
 - ⊕ CCFV provides formal guarantees and more clear extensions
- ▷ E -matching based heuristic instantiation [DNS05; MB07]
 - ⊕ CCFV allows to easily discard instances already entailed by E
- ▷ Model-based instantiation [GM09; RTG+13]
 - ⊕ No need for a secondary ground SMT solver
 - ⊕ No need to guess solutions

Towards a theory solver for instantiation

- ▷ Model generation
- ▷ **Conflict set generation**
- ▷ **Propagation**
- ▷ Incrementality



Congruence Closure with Free Variables

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- ▷ Different possibilities for building solutions are handled with branching and backtracking

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 E \quad \models \quad L\sigma \\
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$$\emptyset \mid$$

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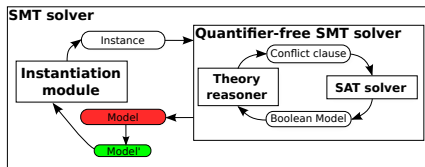
$$z \simeq b \wedge f(x) \simeq f(z)$$

$$y \simeq c, z \simeq b \mid$$

$$f(x) \simeq f(b)$$

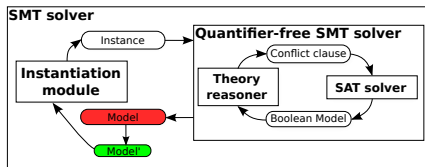
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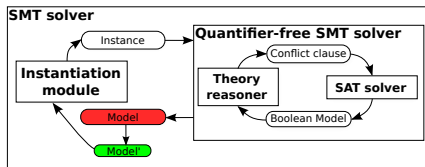


- ▷ Top symbol indexing of E -graph from ground congruence closure

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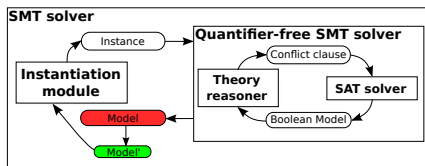


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▶ Bitsets for fast checking if a symbol has applications in a congruence class

Implementation

- ▷ Selection strategies

$$E \models f(x, y) \simeq h(z) \wedge x \simeq t \wedge C$$

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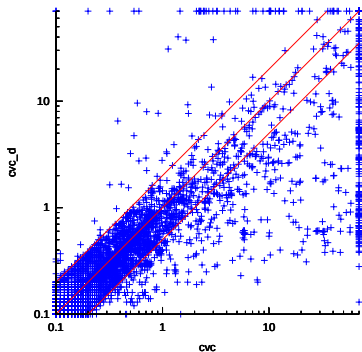
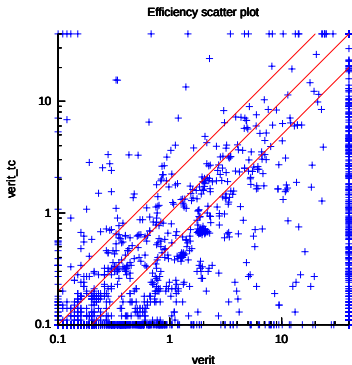
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$$E \models f(t, y) \simeq h(z) \wedge C$$

- ▶ $E \models f(t, y)\sigma \simeq h(z)\sigma$ only if there is some $f(t', t'')$ s.t.

$$E \models t \simeq t'$$



veriT: + 800 out of 1 785 unsolved problems

CVC4: + 200 out of 745 unsolved problems

* experiments in the "UF", "UFLIA", "UFLRA" and "UFIDL" categories of SMT-LIB, which have 10 495 benchmarks annotated as unsatisfiable, with 30s timeout.

Conclusions and future work

- ▷ A unifying framework for quantified formulas with equality and uninterpreted functions
- ▷ Lifting congruence closure to accommodate free variables
- ▷ Efficient implementations in the SMT solvers CVC4 and veriT

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Challenges

- ▷ Finding conflicting instances across multiple quantified formulas

$$E \models \neg\psi_1\sigma \vee \dots \vee \neg\psi_n\sigma, \quad Q = \{\forall\bar{x}. \psi_1, \dots, \forall\bar{x}. \psi_n\}$$

- ▷ Incrementality
- ▷ Learning-based search for solutions
- ▷ Beyond theory of equality
- ▷ Handle variables in E

Thank you

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