DE LA RECHERCHE À L'INDUSTRIE

Real Behavior of Floating Point Numbers

SMT 2017 | Bruno Marre, Bobot François, Zakaria Chihani

23 July 2017

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COLIBRI (Bruno Marre)

- Started in 2000 for test case generation
- Used only as a library in PathCrawler and Gatel
- CP solver uses Eclipse Prolog
- Proprietary with the help of IRSN

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- integer modulo, real

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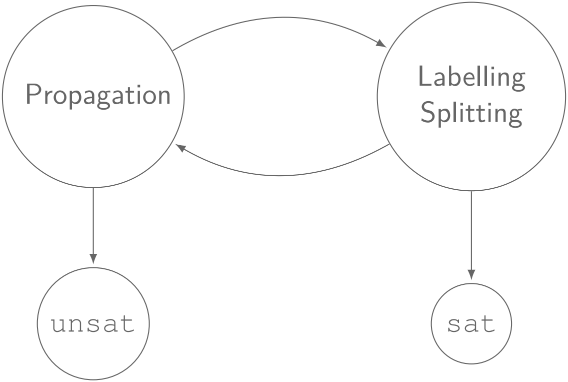
Architecture



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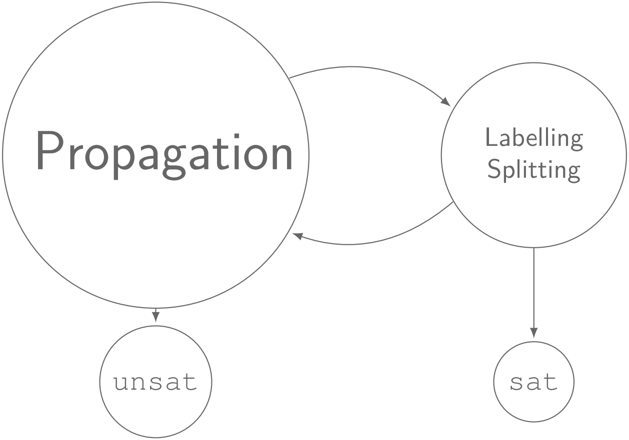

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✔ Clear Semantic: $x \oplus y = \mathrm{o}(x + y)$

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- ✘ Pervasives in programs

Domain Specific Approach of CP

$$X_i \in [1;10] \implies X_0 \oplus X_1 \oplus X_2 \oplus X_3 \oplus X_4 \oplus X_5 \oplus X_6 \oplus X_7 \in [8;80]$$

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Distance graph on floating-point numbers

x	IEEE-format, num(x)
0.	0

$$\mathrm{num}(x) - \mathrm{num}(\texttt{fp.mul\_2}\, x) = 2^{52}$$

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0.	0
$+1p - 1074$	1
$+1p - 1073$	2
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$-1p-1074$	`-1`
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Distance graph on floating-point numbers

$$x \in [1;10],\ \texttt{fp.mul}\ \mathrm{RNE}\ x\ 2 = y$$

$$\begin{tikzcd} x \arrow[rr, bend left, "\{2^{52}\}"] & & y \end{tikzcd}$$

$$w \in [1;10],\ \texttt{fp.add}\ \mathrm{RNE}\ w\ 3 = z$$

$$\begin{tikzcd} w \arrow[rr, bend left, "{[\mathrm{num}(13) - \mathrm{num}(10); \mathrm{num}(4) - \mathrm{num}(1)]}"] & & z \end{tikzcd}$$

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- Instantiated for many functions
- Linearization of constraints for simplex

Interesting and Simple Real Examples

```
/*@ requires 0 ≤ x ≤ 1000;
    requires 0 ≤ y ≤ 1000;
    ensures  0 ≤ \result ≤ 1;   @*/
double x_normalisation(double x,double y){

  return x/sqrt(x*x + y*y);

}
```

COLIBRI: Example of Reasoning

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There is a counter-example!

Interesting and Simple Real Examples: Corrected

```
/*@ requires 0.0001 ≤ x ≤ 1000;
    requires 0.0001 ≤ y ≤ 1000;
    ensures  0 ≤ \result ≤ 1;   @*/
double x_normalisation(double x,double y){

  return x/sqrt(x*x + y*y);

}
```

Other Examples: From SPARK User Rule

```
procedure User_Rule_7 (X, Y, Z, A : Float;
                       Res        : out Boolean)
 is
 begin
    pragma Assume (Z ≥ 0.0);
    pragma Assume (X ≥ Y);
    pragma Assume (Y ≥ Z);
    pragma Assume (X > Z);
    pragma Assume (A ≥ 1.0);
    Res := (X − Y) / (X − Z) ≤ A;
    pragma Assert (Res);      -- valid
 end User_Rule_7;
```

Other Examples: From SPARK User Rule

$$A \leq \frac{X \ominus Y}{X \ominus Z} \leq B \quad \text{with } \ldots$$

$$\sqrt{X^2 \ominus Y^2} \leq X \quad \text{with } \ldots$$

$$\frac{X}{\sqrt{X^2 \oplus Y^2}} \leq 1 \quad \text{with } \ldots$$

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$$(0. \leq_f x \leq_f 10.0) \wedge (0. \leq_f y \leq_f 10.0) \Rightarrow$$
$$((x \oplus y) \ominus x) \ominus y \leq_f 0.0001$$

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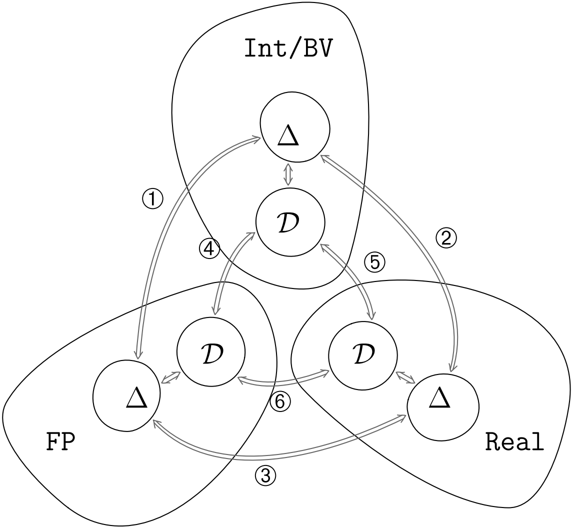
$$(0. \leq_f x \leq_f 10.0) \wedge (0. \leq_f y \leq_f 10.0) \Rightarrow$$
$$\mathsf{o}(\mathsf{o}(\mathsf{o}(x + y) - x) - y) \leq_f 0.0001$$

Bitvector and Integer Arithmetic (CPAIOR17)

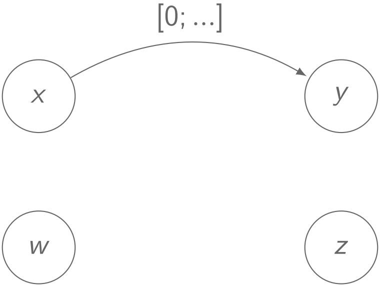
- High-level view of bitvectors
- New propagations for integers $\leftrightarrow$ bitvectors

Interreductions



Casts

$$x, y \in [1; 1000],\ \texttt{fp.to\_sbv}\ \_\ x = w,\ \texttt{fp.to\_sbv}\ \_\ y = z$$



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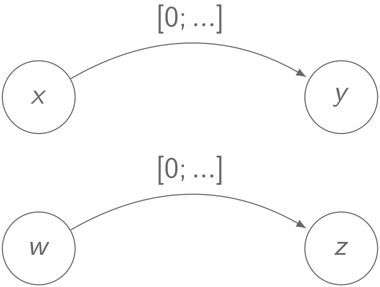
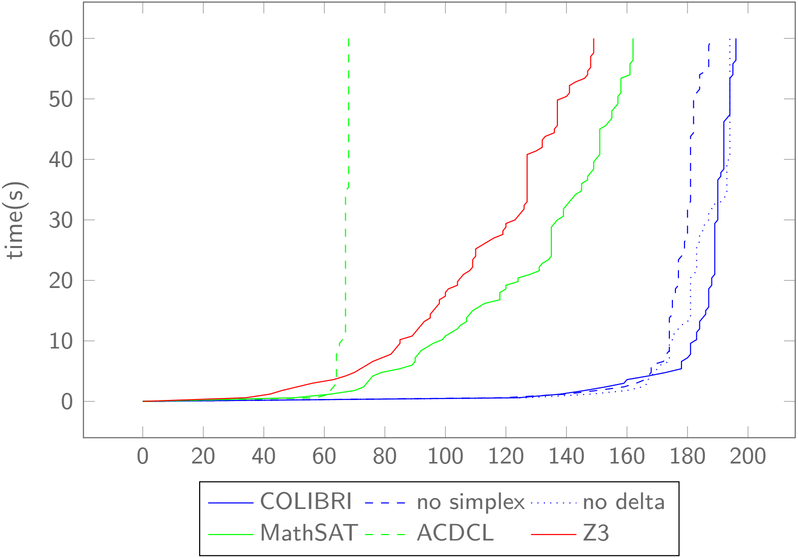


Griggio and Schanda



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- Uninterpreted functions and quantifiers
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- Reduce the loading time...

Theorem

Let $D, E \subset \mathcal{R}$, $f : D \mapsto E$ and $f^{-1} : E \mapsto D$ such that

- $\forall x : D,\ f^{-1}(f(x)) = x$
- *f increasing*

We have

- $\forall x \in D, \mathrm{o}(y) \in E, \mathrm{o}(f(x)) < \mathrm{o}(y) \implies \mathrm{o}(x) \leq \mathrm{o}(f^{-1}(\mathrm{o}(y)))$
- $\forall x \in D, y \in E, \mathrm{o}(f(x)) < \mathrm{o}(f(y)) \implies x < y$

Instantiated for many functions in COLIBRI's DBM

Interesting and Simple Real Examples

```
/*@ ensures \result ≤ (double) 1; @*/
double test2(){
  double x = read_sensor();
  /*@ assert (double) 0 ≤ x ≤ (double) 1000; @*/
  double y = read_sensor();
  double z = read_sensor();

  x = x * x + z * z + y * y + 1;

  if (z ≤ y){
    return (x−y)/(x−z);
  } else {
    return (x−z)/(x−y);
  }
}
```

The Problems