SMT Nonlinear Real Arithmetic and Computer Algebra: a Dialogue of the Deaf?

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Thesis

At a deep level, the problems which SMT's Nonlinear Real Arithmetic (NRA) and Computer Algebra's Cylindrical Algebraic Decomposition (CAD) wish to solve are the same: nevertheless the approaches are completely different, and are described in different languages. We give an NRA/CAD dictionary, explain the CAD process as it is traditionally presented (and some variants), then ask how NRA and CAD might have a more fruitful dialogue.

(partial) Dictionary

Concept	SMT's NRA	CA and CAD
	Arithmetic	Algebra
	Unquantified	$\exists$quantified
	Quantified	Alternation of quantifiers
Goal	A model	Set of all models
	or UNSAT	Quantifier elimination etc.
Starting point	Boolean structure	Polynomials
Order	frequent change	absolutely fixed
	(of boolean variables)	(of theory variables)
Measure	Performance	complexity

Logical/Polynomial Systems over $(\mathbf{R})$

Let p_i be the Boolean $f_i \sigma_i 0$ where $f_i \in \mathbf{Z}[x_1, \ldots, x_n]$ and $\sigma_i \in \{=, \neq, <, \leq, >, \geq\}$.
Let the problem be $\Psi := \mathcal{Q}_1 x_1 \mathcal{Q}_2 x_2 \ldots \mathcal{Q}_n x_n \Phi(p_1, \ldots, p_m)$, where Φ is a Boolean combination (typically in CNF for SAT), and $\mathcal{Q} \in \{\exists, \forall, \text{free}\}$. SMT typically has all $\mathcal{Q}_i$ as $\exists$, QE insists the free occur first (say $x_1, \ldots, x_k$).
Then the goals are:

- **NRA** SAT and a model, or UNSAT (?+proof);
- **CAD** A decomposition of $\mathbf{R}^n$ into D_j such that every f_i is sign-invariant (> 0, $= 0$ or < 0) on each D_j
- **cylindrical** $\forall i, j, k : \pi_k(D_i)$ and $\pi_k(D_j)$ are disjoinnt or equal
- **QE** $\widehat{\Phi}(q_i, \ldots, q_{m'})$, where $q_i := g_i \tau_i 0$, $g_i \in \mathbf{Z}[x_1, \ldots, x_k]$ and $\tau_i \in \{=, \neq, <, \leq, >, \geq\}$.

Approaches (very simplified)

NRA1 Ignore the f_i.

NRA2 Find a Φ-satisfying assigment to p_i.

NRA3 Check this against the theory $p_i = f_i \sigma_i 0$, and SAT

NRA4 or try again (maybe learning a lemma).

QE1 Ignore Φ and the p_i.

QE2 Decompose $\mathbf{R}^n$ into regions (with a sample point) where the f_i are sign-invariant on each region.

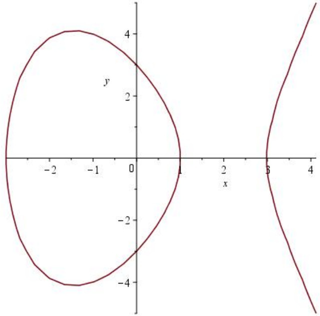
QE3 Evaluate Φ at each sample point.

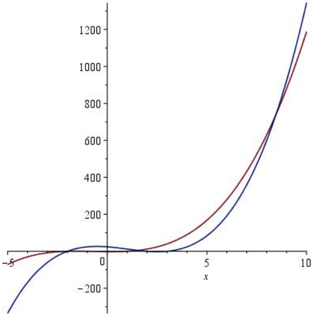
QE4 By cylindricity, evaluate Ψ at sample points of $\mathbf{R}^k$.

$$\forall x_l \Rightarrow \bigwedge_{x_l \text{ sample points}} ; \exists x_l \Rightarrow \bigvee_{x_l \text{ sample points}}$$

QE5 $\widehat{\Phi} := \bigvee$description of Ψ-true cells.

Why might Φ have different values? Geometry($x_2 = y$)





$$\mathrm{disc}_y(y^2 - x^3 + x^2 + 9x - 9)$$

$$4x^3 - 4x^2 - 36x + 36$$

$$\{-3, 1, 3\}$$

$$\mathrm{res}_y(\; y - x^3 - 2x^2 + x + 2,\; y - 2x^3 + 6x^2 + 8x - 24)$$

$$-x^3 + 8x^2 + 7x - 26$$

$$\{-2., 1.535898384, 8.464101616\}$$

So just compute resultants and discriminants?

Not quite: more can go wrong, especially in higher dimensions
We certainly need to worry about contents if non-trivial

- [Col75] Also all coefficients, and subresultants
- [McC84] Not the subresultants
- But a resultant might vanish identically on a set: CAD fails "not well-oriented"
- [Hon90] Unconditional slight improvement on [Col75].
- [Laz94] Conjectures (false proof) we only need leading & trailing coefficients
- [MPP16] Proves Lazard projection (better than McCallum)

So what's the complexity?

Suppose $\Xi_n = \{$ polynomials in $\Phi\}$ has m polynomials of degree $\leq d$ (in each variable).

Then after Geometry(x_n), Ξ_{n-1} has $O(m^2)$ polynomials of degree $O(d^2)$.

Then after Geometry(x_{n-1}), Ξ_{n-2} has $O(m^4)$ polynomials of degree $O(d^4)$.

After Geometry(x_2), Ξ_1 has $m^{2^{O(n)}}$ polynomials of degree $d^{2^{O(n)}}$.

The analysis is significantly messier than this, but qualitatively these results are right.

This doubly-exponential behaviour is inherent in CAD and QE [DH88, BD07], even for the description of a single sample point.

However, for QE these assume $O(n)$ alternations of quantifiers, and there are theoretical results showing $m^{n2^{O(a)}}$, $d^{n2^{O(a)}}$.

But we can do better (by looking at the logic)

SMT It's silly to ignore Φ and p_i.

[Col98] True, if $\Phi = (f_1 = 0) \wedge \Phi'$, we're not interested in Φ' except when $f_1 = 0$.

[McC99] Implemented this: replaces n by $n - 1$ in double exponent of m (therefore $C \to \sqrt{C}$).

- $\Phi := (f_1 = 0 \wedge \Phi_1) \vee (f_2 = 0 \wedge \Phi_2)$ can be written as $f_1 f_2 = 0 \wedge \Phi$ and benefit (but $d \to 2d$)

[BDE+13] address this structure directly

[BDE+16] the case $(f_1 = 0 \wedge \Phi_1) \vee \Phi_2$ etc.

[ED16, DE16] the case $(f_1 = 0) \wedge \cdots \wedge (f_s = 0) \wedge \Phi'$ replaces n by $n - s$ in double exponents of m and d

provided the iterated resultants are primitive: alas not a technicality

Two alternative methods for computing CAD

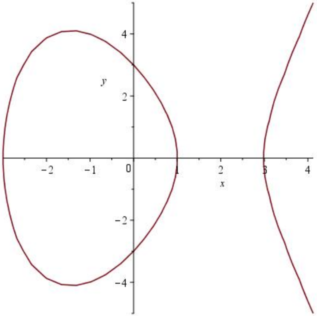
- Regular Chains [CM16]
  1. Decompose $\mathbf{C}^n$ cylindrically by regular chains ($\mathbf{C}^1$ is "special cases" + "the rest")
  2. `MakeSemiAlgebraic` to decompose $\mathbf{R}^i \subset \mathbf{C}^i$ — "the rest" is generally not connected in $\mathbf{R}^i$ and needs to be split up
  3. Read off a CAD
  - Less theory but often better computation in practice
- Comprehensive Gröbner Bases [Wei92]
  1. Build a CGB, i.e. the generic solution *and* all the special cases.
  2. Use this to build CAD [FIS15]
  - Bath have been unable to get this to work

Or Just produce a single cell of the CAD [Bro15]: start from a sample point and see what the obstacles to extending it are

- Inspired by NLSAT [JdM13]

QE Needn't be by CAD: Virtual Term Substitution [Wei98, KS15], very effective for linear/ quadratic problems

There are algebraic deductions: consider



The discriminant is $4x^3 - 4x^2 - 36x + 36$, so $y^2 < x^3 - x^2 - 9x + 9 \Rightarrow (x > -3 \wedge x < 1) \vee (x > 3)$; however $y^2 > x^3 - x^2 - 9x + 9$ gives no deductions.

Does it make sense to partition the logic variables by the theory variables they relate to, and to ask the theory to produce deductions with fewer variables?

More information

- **SC²** Symbolic Computation and Satisfiability Checking. Project description [ABB+16] and www.sc-square.org. Workshop in Kaiserslautern next Saturday and at FLoC 2018.
- **CAD/QE** [CJ98], probably best analysis in [BDE+16].
- **Computer Algebra** [vzGG13] is probably the best text; I am writing one at http://staff.bath.ac.uk/masjhd/JHD-CA.pdf.

Questions?

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