Computing a Complete

Basis for Equalities

Implied by a System of LRA Constraints

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Linear Arithmetic / Linear Programming

Input: $\sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i$ for $i = 1, \ldots, m$

coefficients ($\in \mathbb{Z}$) variables bound ($\in \mathbb{Z}$)

Linear Arithmetic / Linear Programming

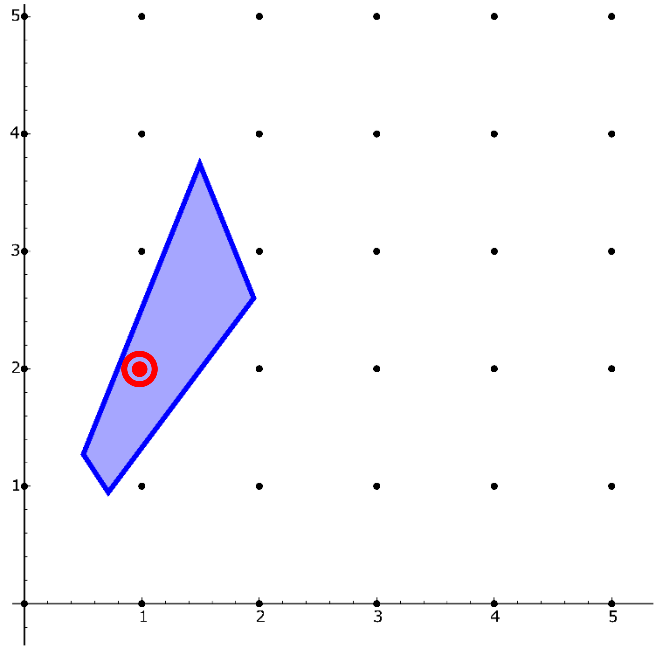
Input: $a_i^T \mathbf{x} \leq b_i$ for $i = 1, \ldots, m$

Goal: $\mathbf{x} \in \mathbb{R}^n$ (LRA/LP) or $\mathbf{x} \in \mathbb{Z}^n$ (LIA/ILP)

Complexity: P NP

Example:

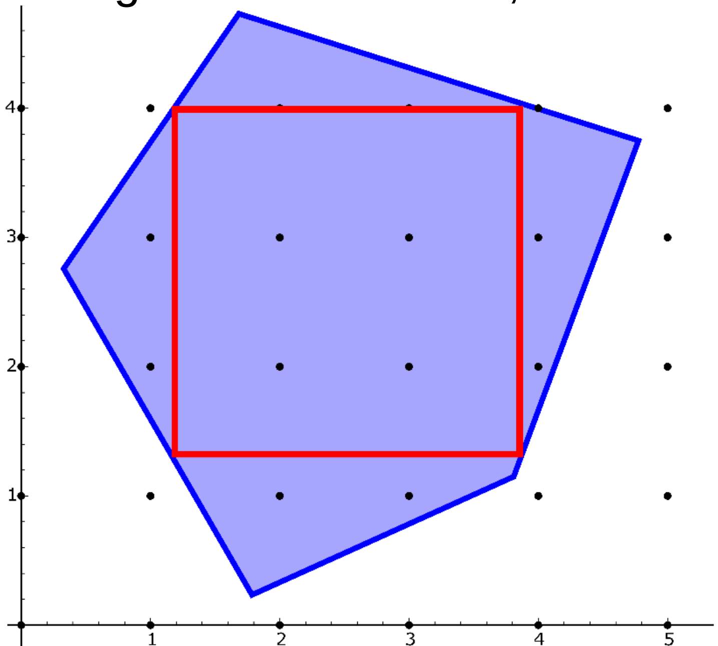
$$2y \leq 5x, \qquad 3y \geq 4x,$$
$$2y \leq -5x + 15, \qquad 2y \geq -3x + 4,$$



LIA: $(x, y) = (1, 2)$

Fast Cube Tests
for LIA constraint solving

(Bromberger & Weidenbach, IJCAR 2016)

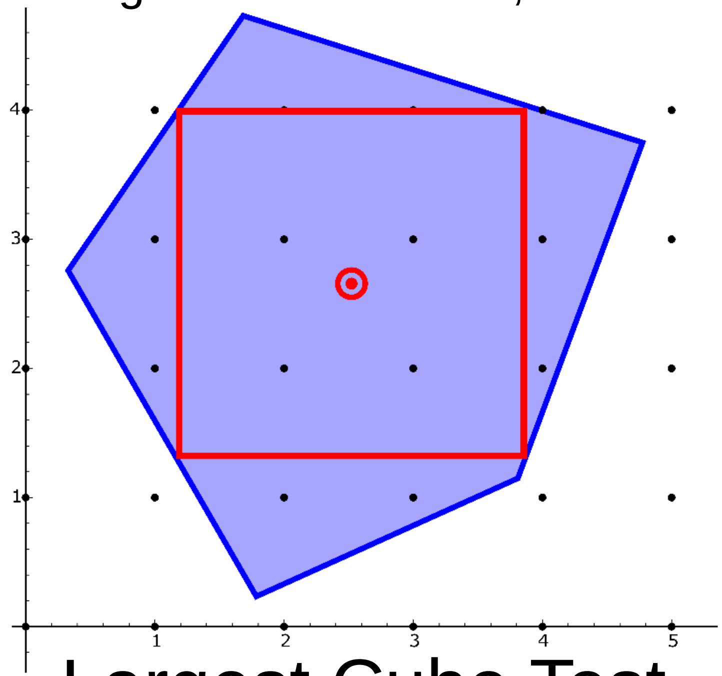


Largest Cube Test

- largest cube inside the set of real solutions

Fast Cube Tests for LIA constraint solving

(Bromberger & Weidenbach, IJCAR 2016)

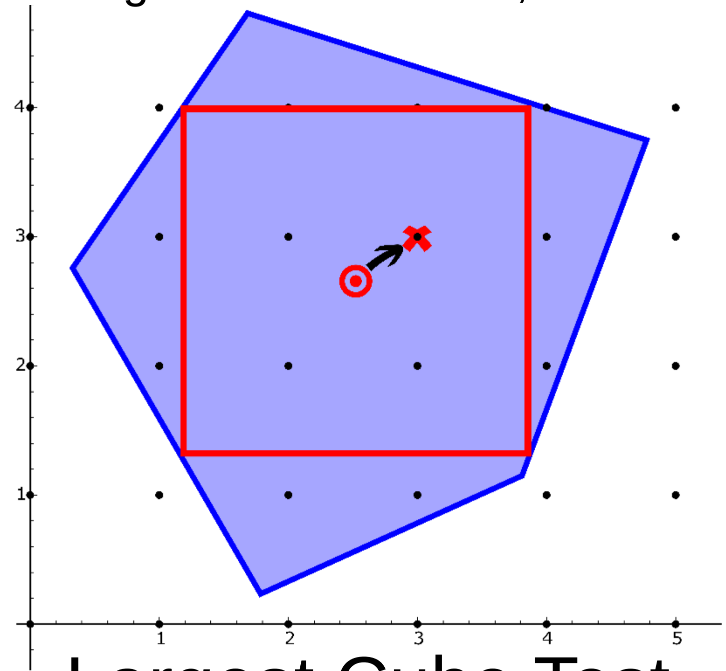


Largest Cube Test

- largest cube inside the set of real solutions
- center point

Fast Cube Tests for LIA constraint solving

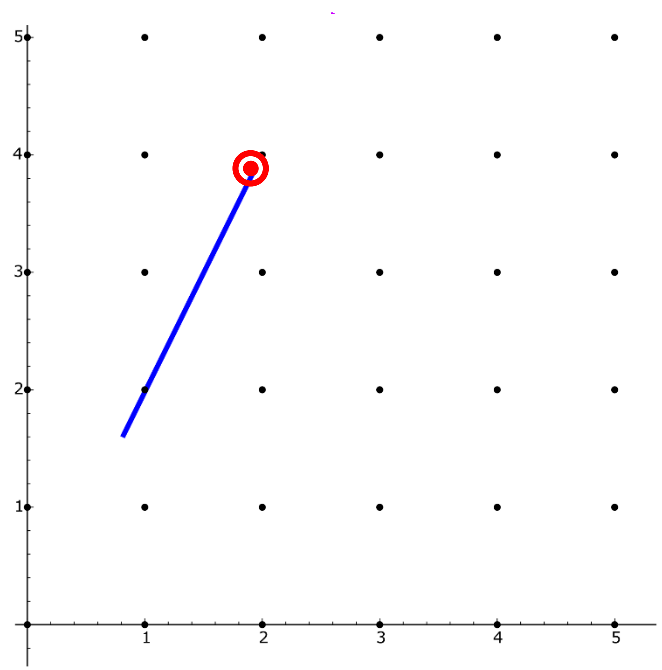
(Bromberger & Weidenbach, IJCAR 2016)

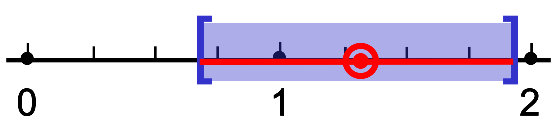


Largest Cube Test

- largest cube inside the set of real solutions
- center point → integer point
- optimization LP (LRA) + evaluation

Equalities





$x \mapsto 1 \;\Rightarrow\; \boxed{y \mapsto 2x} \;\Rightarrow\; y \mapsto 2$

$$2y \leq 2 + 3x, \qquad 2y \geq -2 + 5x,$$
$$5y \leq 25 - 3x, \qquad 2y \geq 4 - 2x,$$
$$\boxed{y = 2x}$$

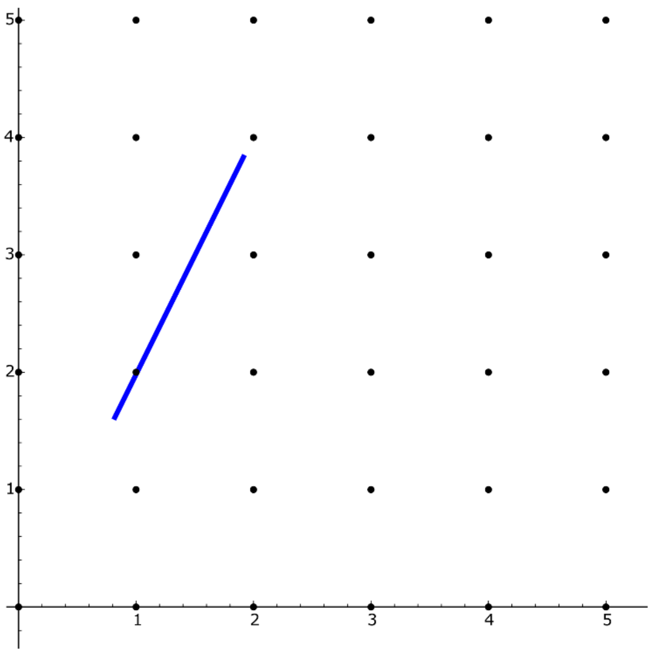
substitute $\Downarrow$ $y \mapsto 2x$

$$2 \cdot 2x \leq 2 + 3x, \qquad 2 \cdot 2x \geq -2 + 5x,$$
$$5 \cdot 2x \leq 25 - 3x, \qquad 2 \cdot 2x \geq 4 - 2x,$$

$\Downarrow$

$$x \leq 2, \qquad 2 \geq x,$$
$$13x \leq 25, \qquad 3x \geq 2,$$

Equalities

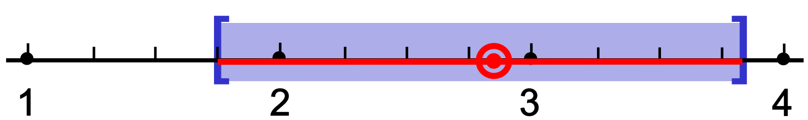


$$2y \le 2 + 3x, \qquad 2y \ge -2 + 5x,$$
$$5y \le 25 - 3x, \qquad 2y \ge 4 - 2x,$$
$$\boxed{y = 2x}$$

Diophantine Equation Handler

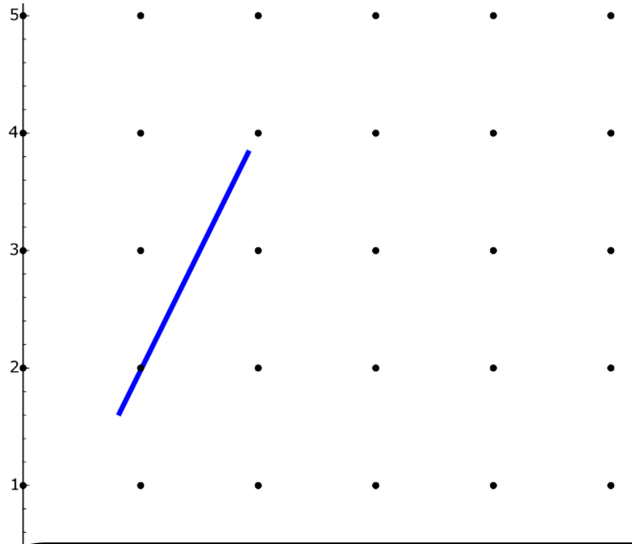
A Practical Approach to
Satisfiability Modulo Linear Integer Arithmetic
by A. Griggio. JSAT 2012

$$y \le 4, \qquad 4 \ge y,$$
$$13y \le 50, \qquad 3y \ge 4,$$



$$y \mapsto 3 \;\Rightarrow\; \boxed{x \mapsto \frac{y}{2}} \;\Rightarrow\; x \mapsto \frac{3}{2}$$

Equalities



$$2y \leq 2 + 3x, \qquad 2y \geq -2 + 5x,$$

$$5y \leq 25 - 3x, \qquad 2y \geq 4 - 2x,$$

$$\boxed{y = 2x}$$

Diophantine Equation Handler

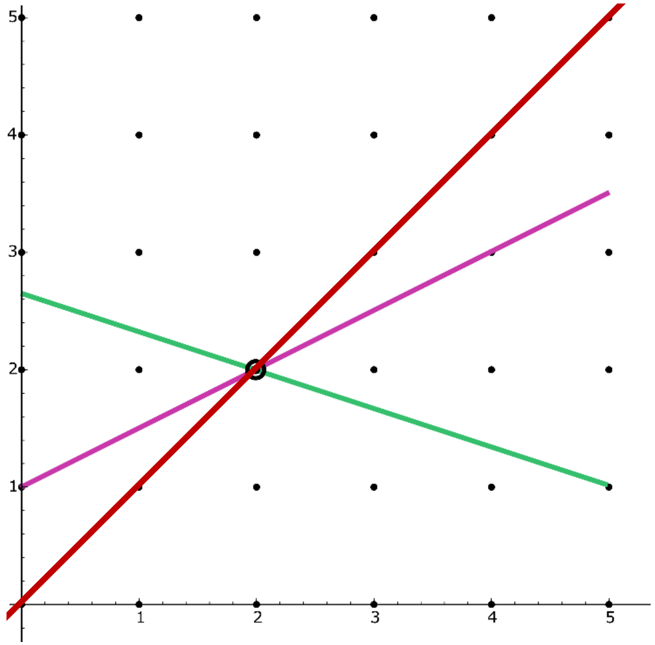
A Practical Approach to

Not Always Explicit!

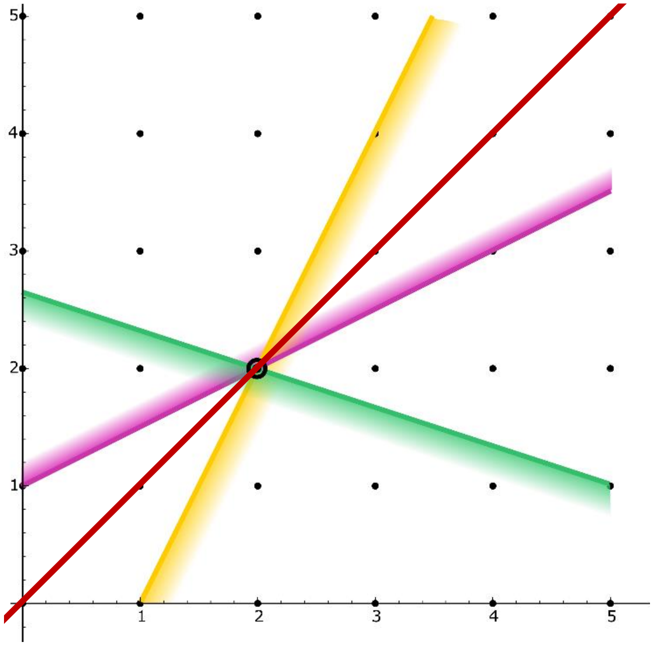
$y \mapsto 3 \Rightarrow \boxed{x \mapsto \frac{y}{2}} \Rightarrow x \mapsto \frac{3}{2}$

$$13y \leq 50, \qquad 3y \geq 4,$$

Implied Equalities



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$



explicit

$$\forall x \in \mathbb{R}^n . \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

implicit

$$\boxed{\boxed{2y = x + 2,} \quad \boxed{3y \leq -x + 8, \qquad 3y \geq -x + 8,}}$$

$$\Downarrow$$

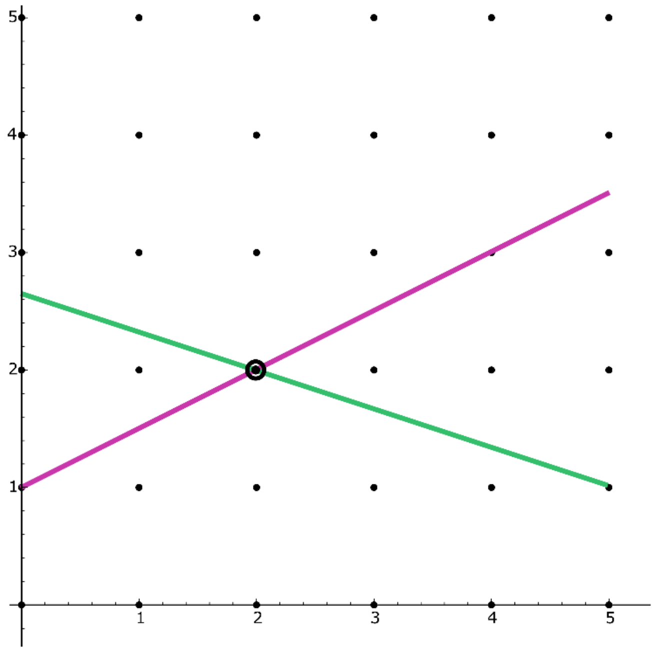
$$\boxed{\boxed{2y = x + 2} \quad \boxed{3y = -x + 8} \quad \boxed{y = 2x - 2,}}$$

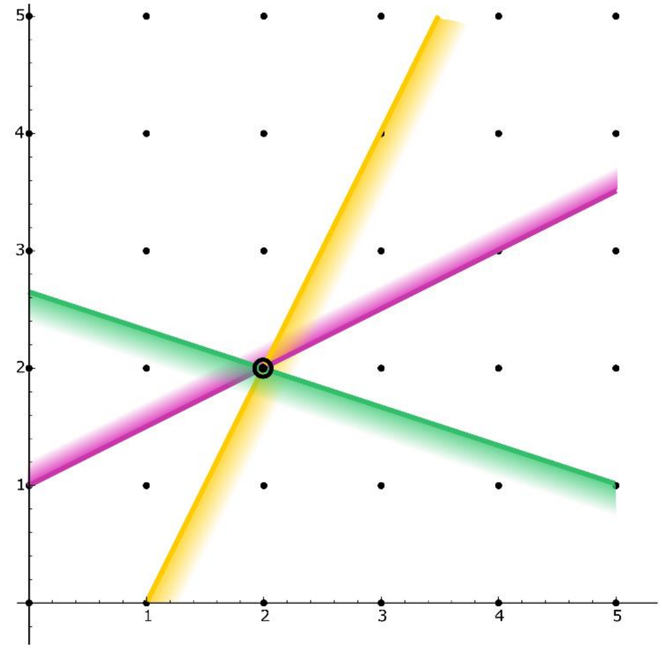
$$\boxed{\boxed{2y \geq x + 2,} \quad \boxed{3y \leq -x + 8,} \quad \boxed{y \leq 2x - 2,}}$$

$$\Downarrow$$

$$\boxed{\boxed{2y = x + 2} \quad \boxed{3y = -x + 8} \quad \boxed{y = 2x - 2,}}$$

Equality Basis





explicit

$$\forall x \in \mathbb{R}^n . \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

implicit

$$2y = x + 2, \quad 3y \leq -x + 8, \quad 3y \geq -x + 8,$$

$$\Downarrow$$

$$2y = x + 2 \quad 3y = -x + 8 \quad y = 2x - 2,$$

$$2y \geq x + 2, \quad 3y \leq -x + 8, \quad y \leq 2x - 2,$$

$$\Downarrow$$

$$2y = x + 2 \quad 3y = -x + 8 \quad y = 2x - 2,$$

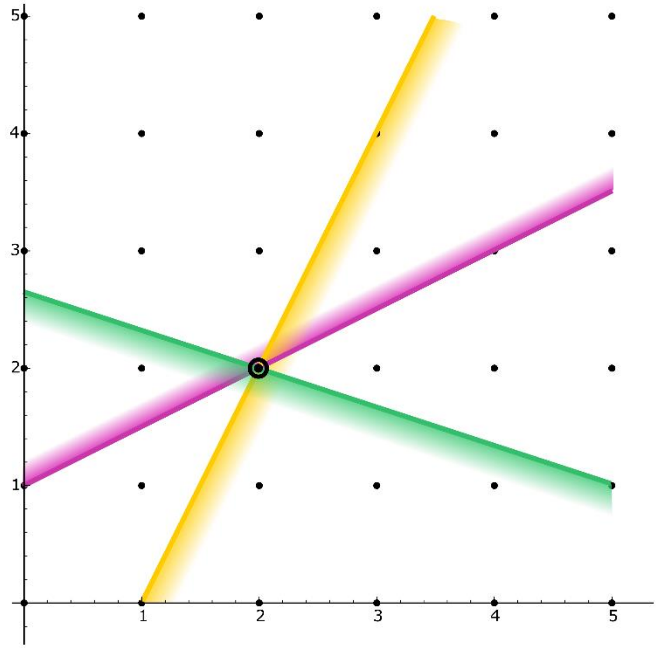
Equality Basis

Equality Basis:

1.set of

linear independent equalities

2. maximal



Implied Equalites:

All

linear combinations

of the

Equality Basis

$$\forall x \in \mathbb{R}^n . \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

$$\boxed{2y \geq x + 2,}$$

$$\boxed{3y \leq -x + 8,} \quad \boxed{y \leq 2x - 2,}$$

$$\Downarrow$$

$$\boxed{2y = x + 2} \quad \boxed{3y = -x + 8} \quad \boxed{y = 2x - 2,}$$

$$\lambda_1, \lambda_2 \in \mathbb{R}$$

$$\lambda_1 \cdot \boxed{2y = x + 2} + \lambda_2 \cdot \boxed{3y = -x + 8}$$

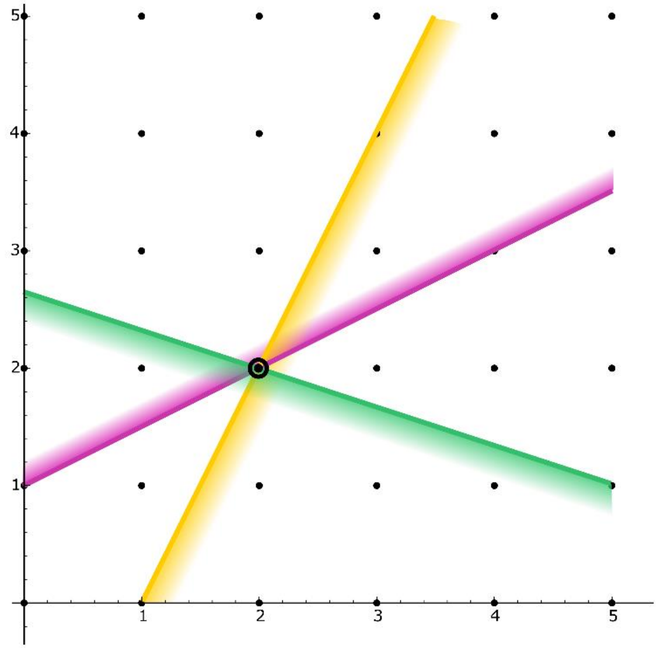
$$\Downarrow$$

$$(2\lambda_1 + 3\lambda_2)y = (\lambda_1 - \lambda_2)x + (2\lambda_1 - 8\lambda_2)$$

Equality Basis

Equality Basis:

1. set of linear independent equalities
2. maximal



Implied Equalites:

All linear combinations of the Equality Basis

$$\forall x \in \mathbb{R}^n . \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

$$\boxed{2y \geq x + 2,}$$

$$\boxed{3y \leq -x + 8,} \quad \boxed{y \leq 2x - 2,}$$

$$\Downarrow$$

$$\boxed{2y = x + 2}$$

$$\boxed{3y = -x + 8}$$

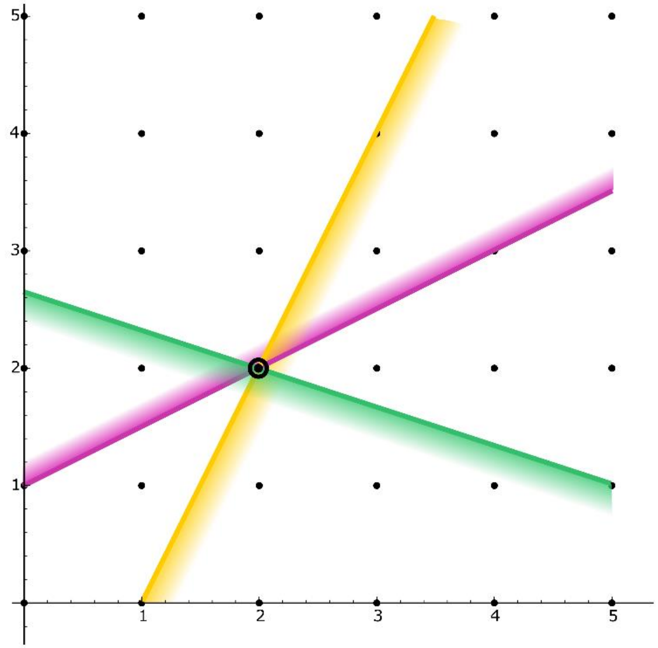
$$\frac{7}{5} \cdot \boxed{2y = x + 2} - \frac{3}{5} \cdot \boxed{3y = -x + 8}$$

$$\Downarrow$$

$$\boxed{y = 2x - 2,}$$

Equality Basis

Simplification:
Eliminating
Equalities
via
Substitution



$$\forall x \in \mathbb{R}^n . \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

$2y \geq x + 2,$

$3y \leq -x + 8,$ $\quad y \leq 2x - 2,$

$2y = x + 2$

$3y = -x + 8$

$2y = x + 2$

$x \mapsto 2y - 2$

$3y = -x + 8$

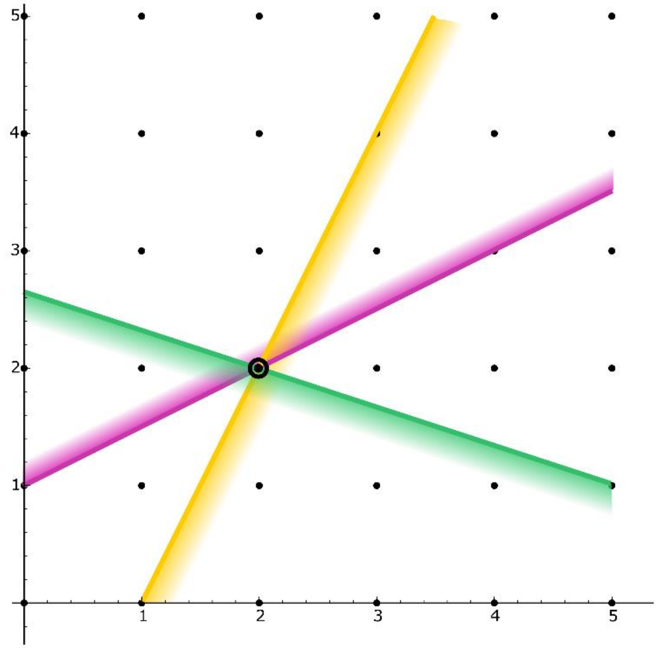
$5y = 10$

$y \mapsto 2$

Equality Basis

Simplification:

Eliminating Equalities via Substitution



Verifying Implied Equalities:

via Substitution (Nelson-Oppen)

$$\forall x \in \mathbb{R}^n . \boxed{Ax \leq b} \rightarrow \boxed{d^T x = c}$$

$$2y \geq x + 2, \quad 3y \leq -x + 8, \quad y \leq 2x - 2,$$

$$\Downarrow$$

$$2y = x + 2 \qquad 3y = -x + 8$$

?

$$y = 2x - 2$$

$$x \mapsto 2y - 2$$

$$\Downarrow$$

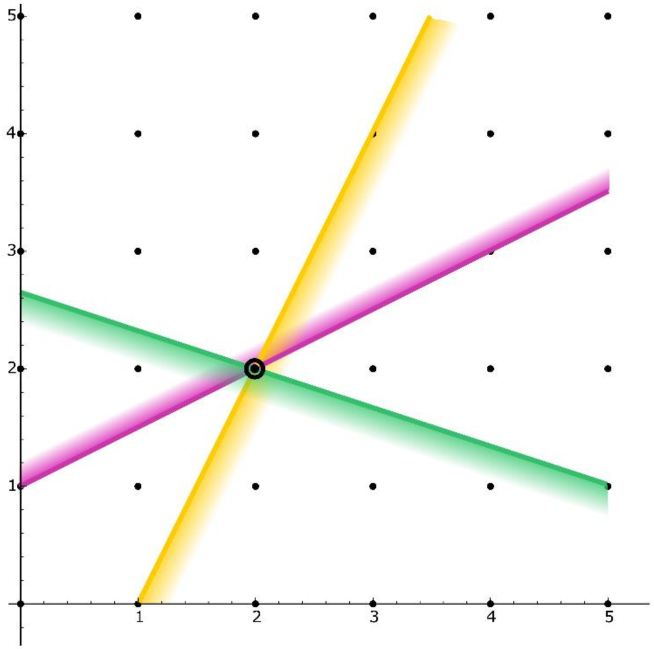
$$y = 4y - 6$$

$$y \mapsto 2$$

$$\Downarrow$$

$$0 = 0$$

Nelson-Oppen Combination



$$u = v$$

pairs of equivalent variables

$$\mathcal{T}$$

$$w = z$$

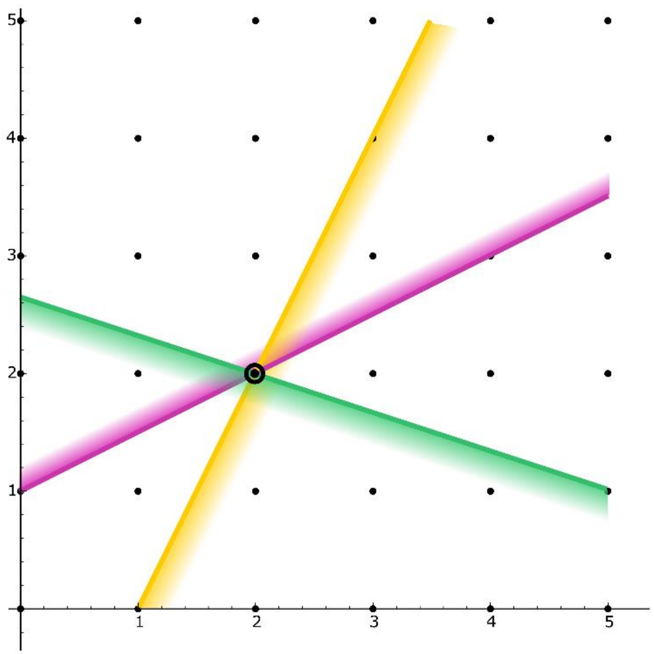
$$\sigma := \{\boxed{x \mapsto 2y - 2}, \boxed{y \mapsto 2}\}$$

$$x\sigma = 2$$

$$y\sigma = 2$$

$$\Longrightarrow \quad x = y$$

Nelson-Oppen Combination



$u = v$

pairs of equivalent variables

$\mathcal{T}$

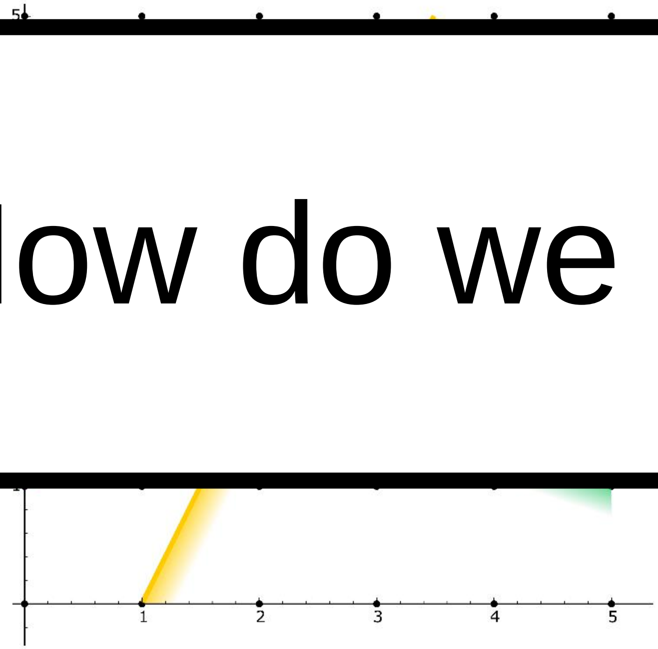
$w = z$

semantic equivalence $\xRightarrow{\sigma\text{ + normalizing}}$ syntactic equivalence

Find equivalent variables with DAGs!

Nelson-Oppen Combination

$$u = v$$



How do we find equalities?

$$w = z$$

semantic equivalence $\xrightarrow{\sigma \text{ + normalizing}}$ syntactic equivalence

Find equivalent variables with DAGs!

Largest Cube Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

$\Rightarrow$

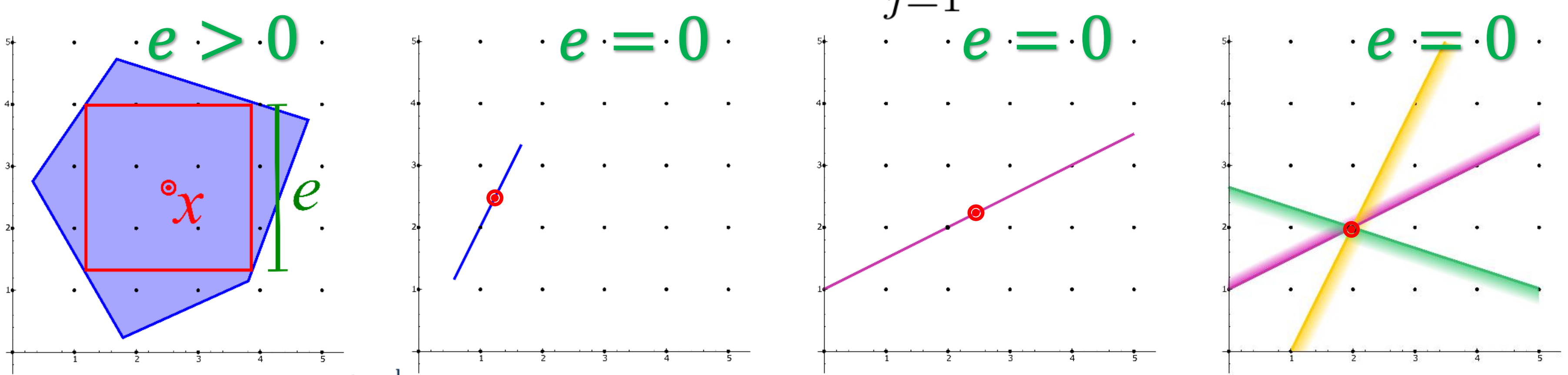
maximize e

subject to:

$$a_i^T x + \|a_i\|_1 \frac{e}{2} \leq b_i$$

for $i = 1, \ldots, m$

$$\text{where } \|a_i\|_1 := \sum_{j=1}^{n} |a_{ij}|$$



Equality Test

subject to:

$a_i^T x \leq b_i$

for $i = 1, \ldots, m$

$\Rightarrow$

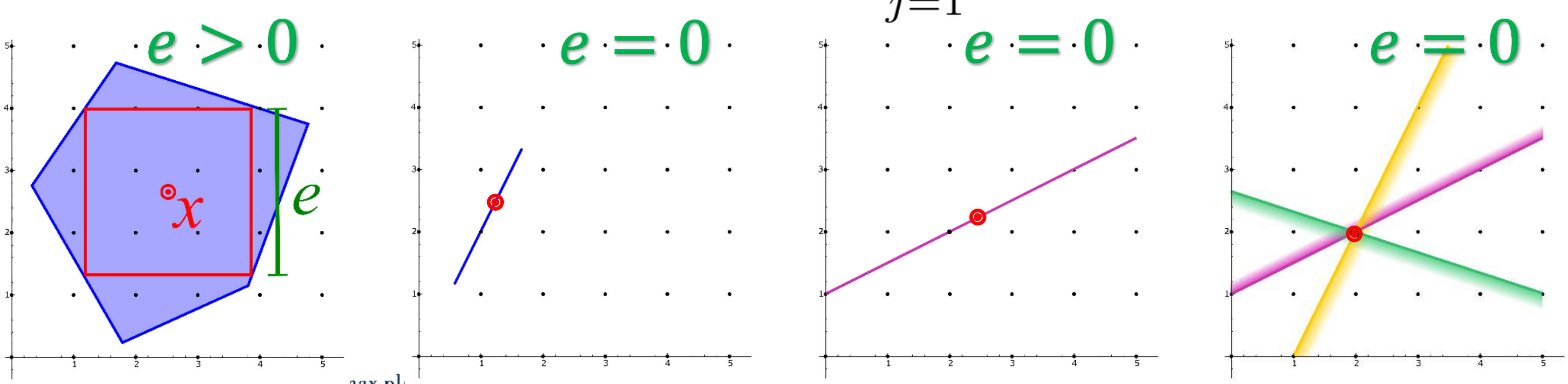
$e > 0$

subject to:

$a_i^T x + e \leq b_i$

for $i = 1, \ldots, m$

$$\text{where } \|a_i\|_1 := \sum_{j=1}^{n} |a_{ij}|$$



Equality Test

subject to:

$a_i^T x \leq b_i$

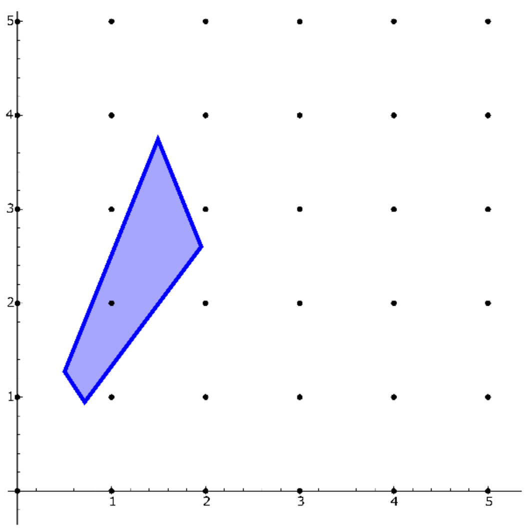
for $i = 1, \ldots, m$

subject to:

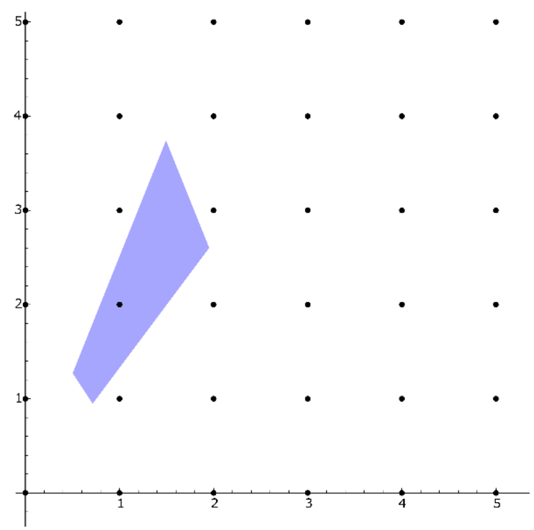
$a_i^T x < b_i$

for $i = 1, \ldots, m$

Interior & Surface



Interior



Equality Test

subject to:

$$a_i^T x \leq b_i$$

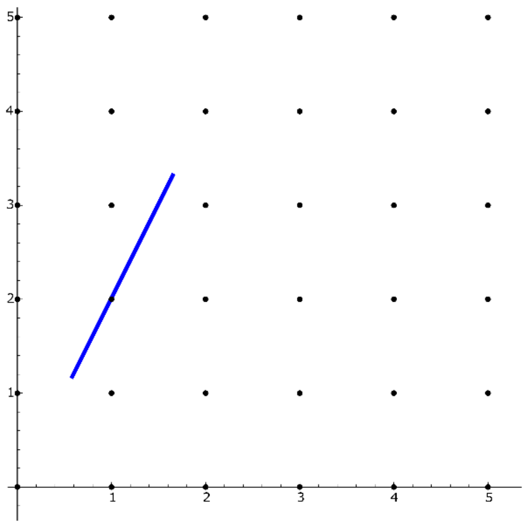
for $i = 1, \ldots, m$

subject to:

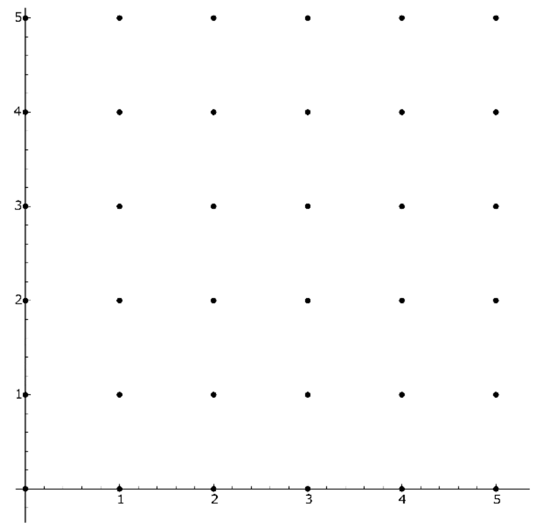
$$a_i^T x < b_i$$

for $i = 1, \ldots, m$

Interior & Surface



Interior



Equality Test

subject to:

$$a_i^T x \leq b_i$$

for $i = 1, \ldots, m$

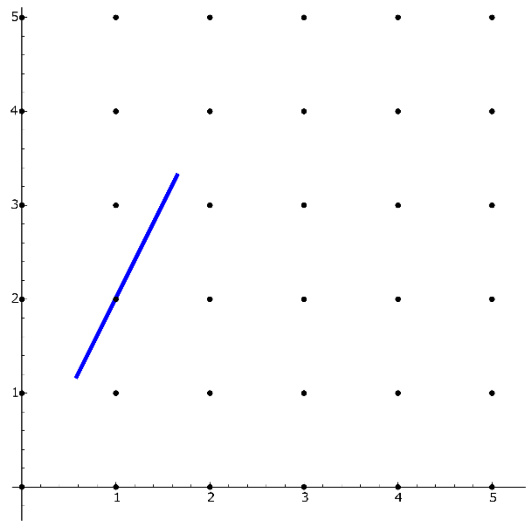
sat

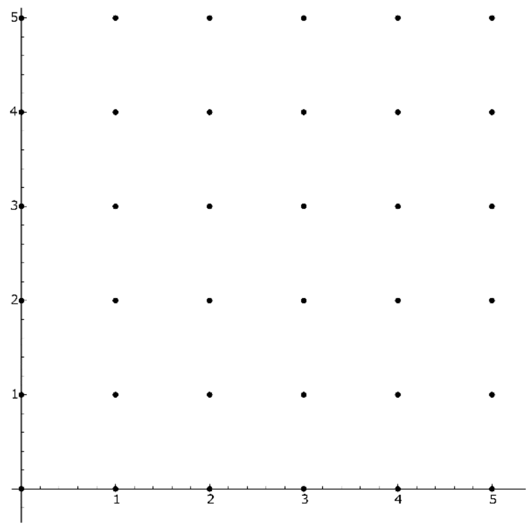
subject to:

$$a_i^T x < b_i$$

for $i = 1, \ldots, m$

unsat





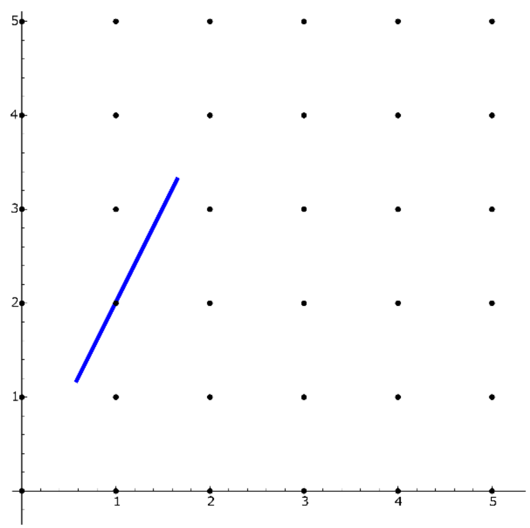
Equality Test

subject to:

$$a_i^T x \leq b_i$$

$$\text{for } i = 1, \ldots, m$$

sat



subject to:

$$a_i^T x < b_i$$

$$\text{for } i = 1, \ldots, m$$

unsat

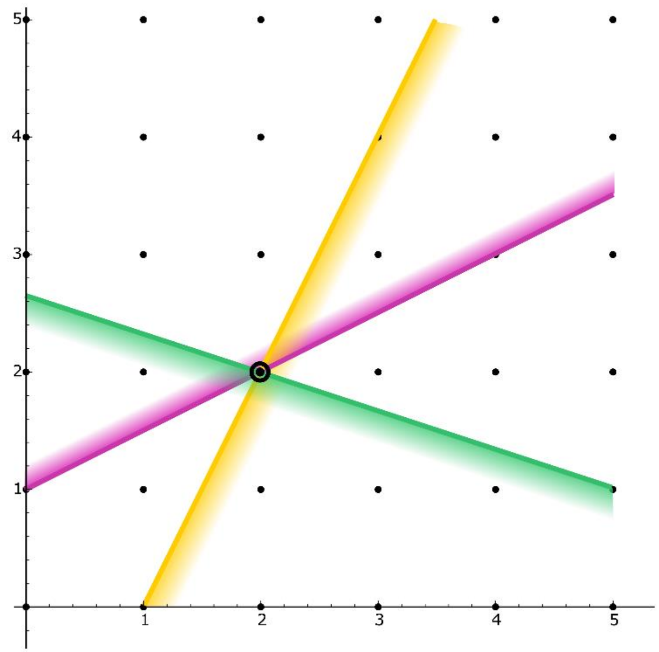
$$\text{Conflict: } C \subseteq \{1, \ldots, m\}$$

$$a_i^T x < b_i \quad \text{for } i \in C$$

unsat

$$a_i^T x = b_i \quad \text{for } i \in C$$

Equality Explanation



original

$$-2x + y \le -2,$$
$$x + 3y \le 8,$$
$$x - 2y \le -2,$$

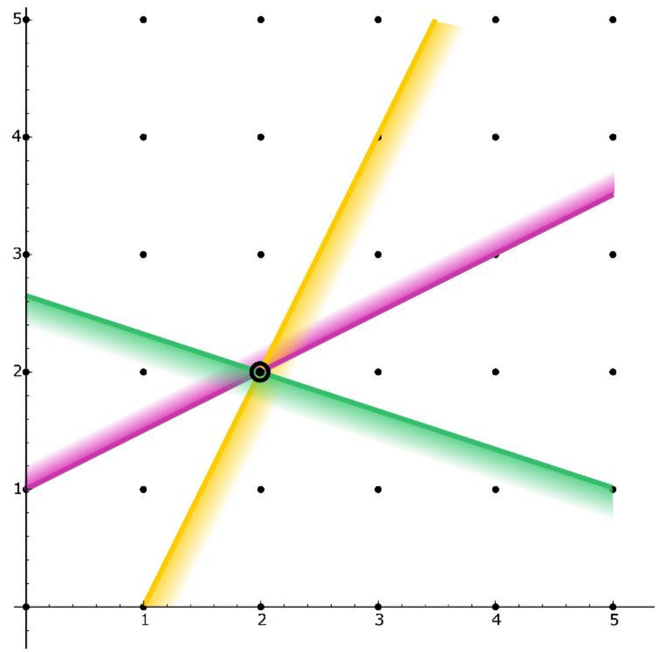
$\Rightarrow$ Positive Linear Combinations

$$-2x + y \le -2,$$
$$2x - y \le 2,$$

$\Rightarrow$

$$-2x + y = -2,$$

Equality Explanation



original

$-2x + y \leq -2,$ $\quad \cdot 1 \quad \rightarrow \quad$ $-2x + y \leq -2,$

$x + 3y \leq 8,$ $\quad \cdot \frac{3}{5} \quad \rightarrow \quad$ $\frac{3}{5}x + \frac{9}{5}y \leq \frac{24}{5},$

$x - 2y \leq -2,$ $\quad \cdot \frac{7}{5} \quad \rightarrow \quad$ $\frac{7}{5}x - \frac{14}{5}y \leq -\frac{14}{5},$

$+ \quad \rightarrow \quad$ $2x - y \leq \quad 2,$

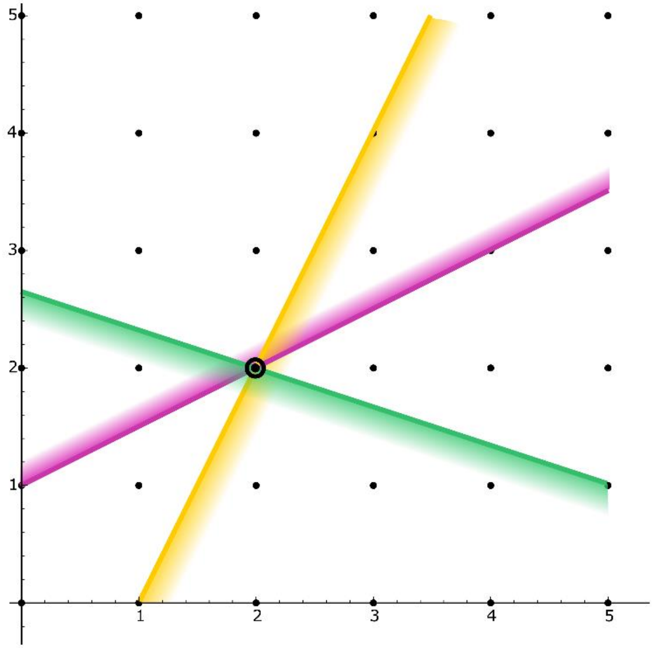
Equality Explanation

strict

$-2x + y < -2,$

$x + 3y < 8,$

$x - 2y < -2,$



original

$-2x + y \leq -2,$ $\cdot 1$ $\Rightarrow$ $-2x + y \leq -2,$

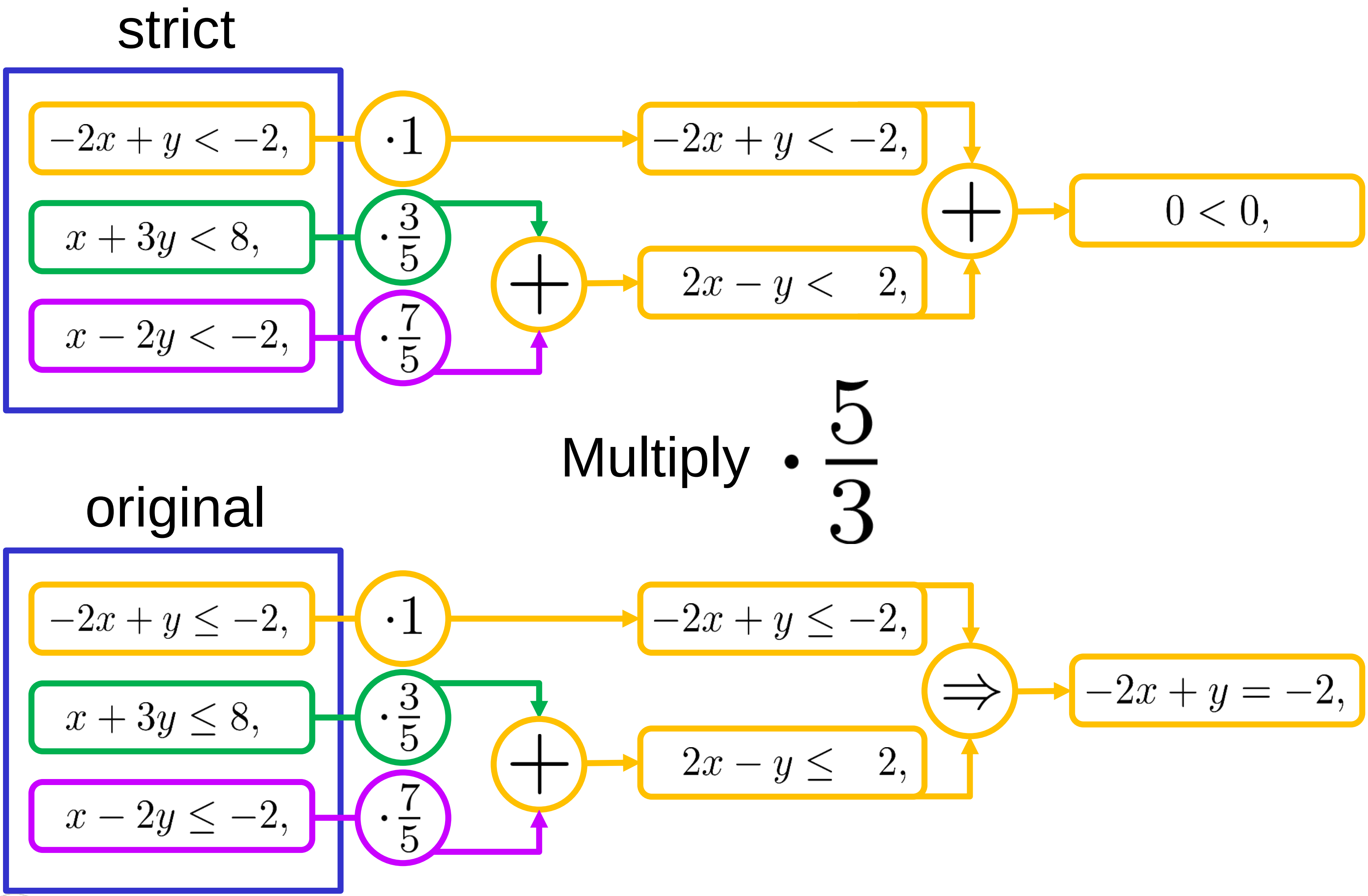
$x + 3y \leq 8,$ $\cdot \frac{3}{5}$

$x - 2y \leq -2,$ $\cdot \frac{7}{5}$

$+$ $\Rightarrow$ $2x - y \leq \quad 2,$

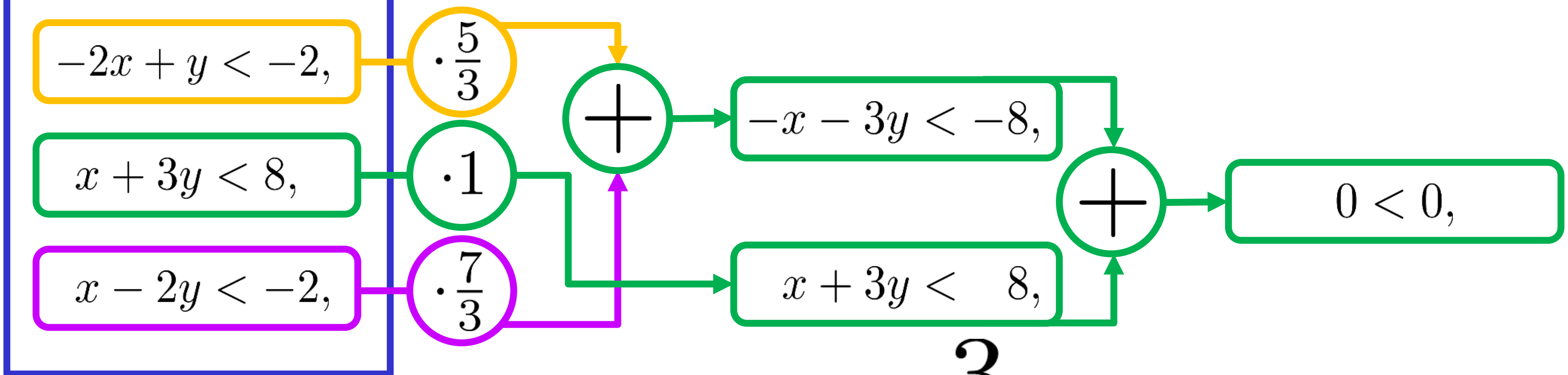
$\Rightarrow$ $-2x + y = -2,$

Equality Explanation



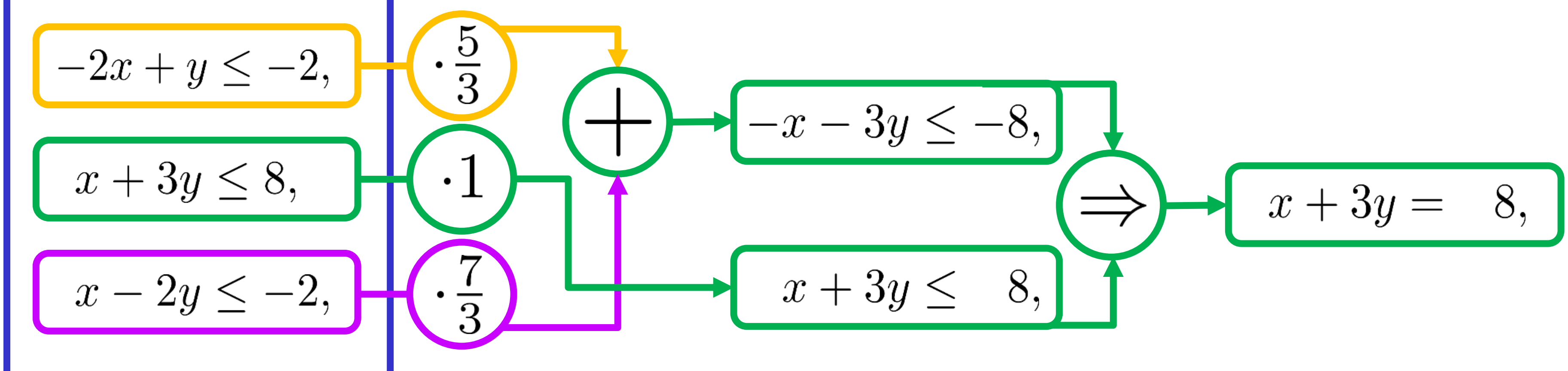
Equality Explanation

strict

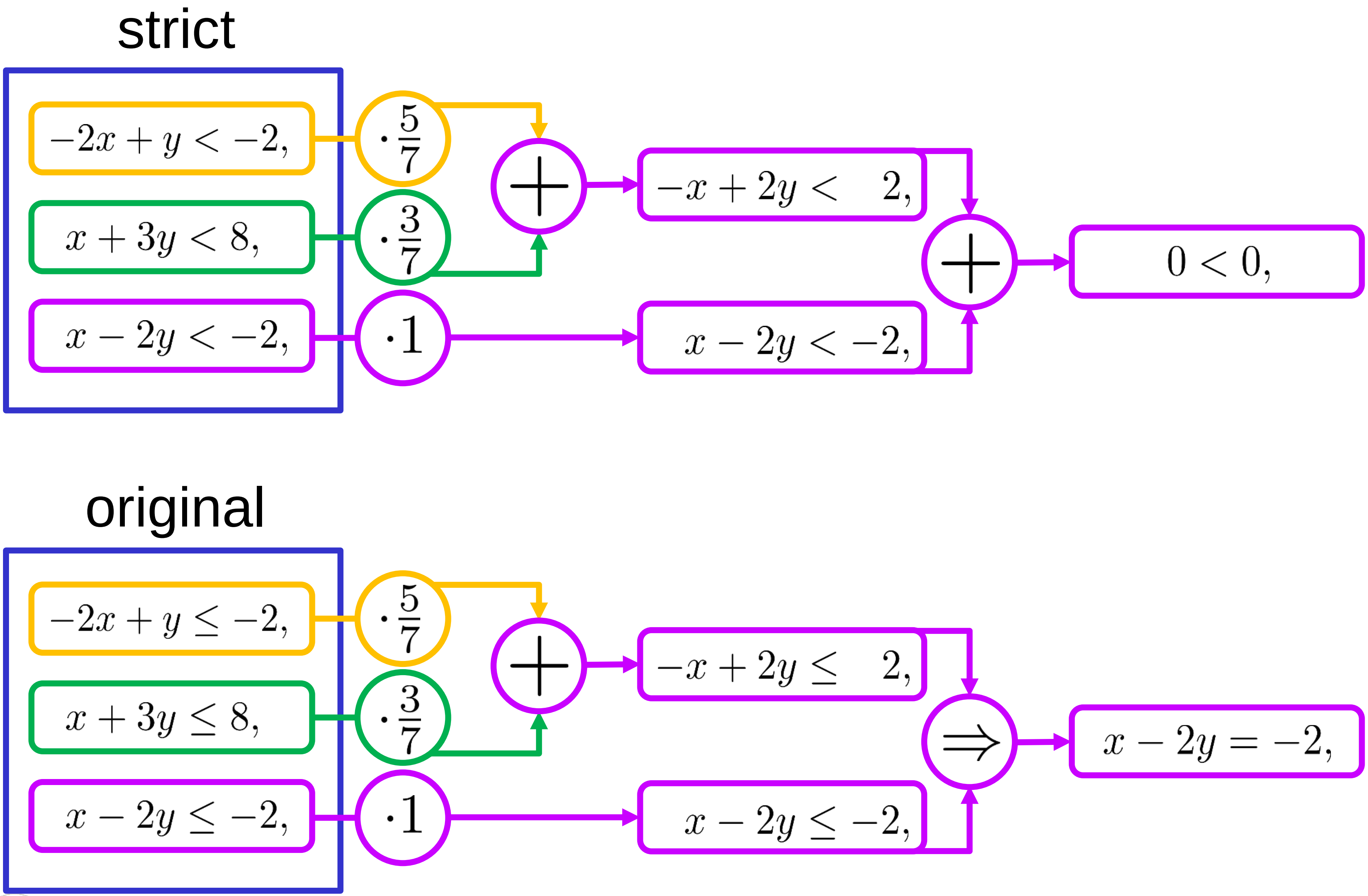


Multiply $\cdot \frac{3}{7}$

original



Equality Explanation



Equality Test

Original System of Constraints $\mathcal{P}$

actual inequalities	implying inequalities	strict inequalities
$a_i^T\mathbf{x} \leq b_i$	$a_i^T\mathbf{x} \leq b_i$	$a_i^T\mathbf{x} < b_i$
$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$	$\mathcal{P} \rightarrow a_i^T\mathbf{x} = b_i$	$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$
	Equality Explanation	

$a_i^T\mathbf{x} < b_i$	$a_i^T\mathbf{x} < b_i$	$a_i^T\mathbf{x} < b_i$
	Minimal Conflict	

System for Equality Test

Equality Test

Original System of Constraints $\mathcal{P}$

actual inequalities	implying inequalities	strict inequalities
$a_i^T \mathbf{x} \leq b_i$	$a_i^T \mathbf{x} \leq b_i$	$a_i^T \mathbf{x} < b_i$
$\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$	$\mathcal{P} \rightarrow a_i^T \mathbf{x} = b_i$	$\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$
	Equality Explanation	

$a_i^T \mathbf{x} < b_i$	$a_i^T \mathbf{x} < b_i$	
	Minimal Conflict	

System for Equality Test

Equality Test

Original System of Constraints $\mathcal{P}$

non-strict inequalities		strict inequalities
$a_i^T\mathbf{x} \leq b_i$	$a_i^T\mathbf{x} \leq b_i$	$a_i^T\mathbf{x} < b_i$
$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$	$\mathcal{P} \rightarrow a_i^T\mathbf{x} = b_i$	$\mathcal{P} \not\rightarrow a_i^T\mathbf{x} = b_i$

$a_i^T\mathbf{x} < b_i$	$a_i^T\mathbf{x} < b_i$

System for Equality Test

Equality Test

Original System of Constraints $\mathcal{P}$		
non-strict inequalities $\boxed{y \geq 2x,}$		strict inequalities
$a_i^T \mathbf{x} \leq b_i$ $\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$ $\boxed{2y \geq 2v - x,}$	$a_i^T \mathbf{x} \leq b_i$ $\mathcal{P} \rightarrow a_i^T \mathbf{x} = b_i$	$a_i^T \mathbf{x} < b_i$ $\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$

Satisfiable Assignment for $\mathcal{P}$

$$x \mapsto 1 \quad y \mapsto 2 \quad v \mapsto 1$$

$$\boxed{2y \geq 2v - x,} \Rightarrow \boxed{2 \cdot 2 \geq 2 \cdot 1 - 1,} \Rightarrow \boxed{4 \geq 3,}$$

$$\boxed{y \geq 2x,} \Rightarrow \boxed{2 \geq 2 \cdot 1,} \Rightarrow \boxed{2 \geq 2,}$$

Equality Test

Original System of Constraints $\mathcal{P}$

non-strict inequalities $\boxed{y \geq 2x,}$		strict inequalities
$a_i^T \mathbf{x} \leq b_i$	$a_i^T \mathbf{x} \leq b_i$	$a_i^T \mathbf{x} < b_i$
$\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$	$\mathcal{P} \rightarrow a_i^T \mathbf{x} = b_i$	$\mathcal{P} \not\rightarrow a_i^T \mathbf{x} = b_i$
$\boxed{2y \geq 2v - x,}$		

$\Downarrow$ $\Downarrow$ $\Downarrow$

$a_i^T \mathbf{x} < b_i$	$a_i^T \mathbf{x} < b_i$
$\boxed{y > 2x,}$	

System for Equality Test

Computing an Equality Basis

original

$$
\begin{aligned}
2u &\geq 2 + v - x, \\
3u &\leq 8 - v - 4x, \\
u &\leq -2 + 2v + x, \\
2y &\leq 9 + 3u - 2x, \\
2y &\geq 2v - x, \\
y &\leq 4 - v - u, \\
y &\geq 2x,
\end{aligned}
$$

sat

strict

$$
\boxed{
\begin{aligned}
2u &> 2 + v - x, \\
3u &< 8 - v - 4x, \\
u &< -2 + 2v + x,
\end{aligned}
}
$$

$$
\begin{aligned}
2y &< 9 + 3u - 2x, \\
2y &> 2v - x, \\
y &< 4 - v - u, \\
y &> 2x,
\end{aligned}
$$

unsat

equalities

$$2u = 2 + v - x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$

Computing an Equality Basis

original

$$\begin{aligned} 2u &\geq 2 + v - x, \\ 3u &\leq 8 - v - 4x, \\ u &\leq -2 + 2v + x, \\ 2y &\leq 9 + 3u - 2x, \\ 2y &\geq 2v - x, \\ y &\leq 4 - v - u, \\ y &\geq 2x, \end{aligned}$$

sat

strict

$$\boxed{\begin{aligned} 0 &> 0, \\ 5u &< 10 - 5x, \\ -3u &< -6 + 3x, \end{aligned}}$$

$$\begin{aligned} 2y &< 9 + 3u - 2x, \\ 2y &> -4 + 4u + x, \\ y &< 6 - 3u - x, \\ y &> 2x, \end{aligned}$$

unsat

equalities

$$2u = 2 + v - x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

strict

$$\boxed{\begin{aligned} 5u &< 10 - 5x, \\ -3u &< -6 + 3x, \end{aligned}}$$
$$2y < 9 + 3u - 2x,$$
$$2y > -4 + 4u + x,$$
$$y < 6 - 3u - x,$$
$$y > 2x,$$

unsat

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$

Computing an Equality Basis

original

$$\begin{aligned} 2u &\geq 2 + v - x, \\ 3u &\leq 8 - v - 4x, \\ u &\leq -2 + 2v + x, \\ 2y &\leq 9 + 3u - 2x, \\ 2y &\geq 2v - x, \\ y &\leq 4 - v - u, \\ y &\geq 2x, \end{aligned}$$

sat

strict

$$\boxed{\begin{aligned} 0 &< 0, \\ 0 &< 0, \end{aligned}}$$

$$\begin{aligned} 2y &< 15 - 5x, \\ 2y &> 4 - 3x, \\ y &< 2x, \\ y &> 2x, \end{aligned}$$

unsat

equalities

$$\begin{aligned} 2u &= 2 + v - x, \\ 5u &= 10 - 5x, \end{aligned}$$

substitutions

$$\begin{aligned} v &\mapsto -2 + 2u + x, \\ u &\mapsto 2 - x, \end{aligned}$$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

strict

$$2y < 15 - 5x,$$
$$2y > 4 - 3x,$$
$$y < 2x,$$
$$y > 2x,$$

unsat

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$
$$y = 2x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$
$$y \mapsto 2x,$$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

strict

$$9x < 15,$$
$$7x > 4,$$
$$\boxed{0 < 0, \quad 0 > 0,}$$

unsat

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$
$$y = 2x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$
$$y \mapsto 2x,$$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

$\Rightarrow$

strict

$$9x < 15,$$
$$7x > 4,$$

sat

$\Rightarrow$

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$
$$y = 2x,$$

$\Downarrow$ **substitutions** $\Downarrow$

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$
$$y \mapsto 2x,$$

$\Leftarrow$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

simplified

$$9x \leq 15,$$
$$7x \geq 4,$$

sat

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$
$$y = 2x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$
$$y \mapsto 2x,$$

Computing an Equality Basis

original

$$2u \geq 2 + v - x,$$
$$3u \leq 8 - v - 4x,$$
$$u \leq -2 + 2v + x,$$
$$2y \leq 9 + 3u - 2x,$$
$$2y \geq 2v - x,$$
$$y \leq 4 - v - u,$$
$$y \geq 2x,$$

sat

simplified

$$9x \leq 15,$$
$$7x \geq 4,$$

sat

equalities

$$2u = 2 + v - x,$$
$$5u = 10 - 5x,$$
$$y = 2x,$$

substitutions

$$v \mapsto -2 + 2u + x,$$
$$u \mapsto 2 - x,$$
$$y \mapsto 2x,$$

$$u \stackrel{?}{=} v \Rightarrow u = -2 + 2u + x \Rightarrow 2 - x = 2 - x \Rightarrow 0 = 0$$

Nelson-Oppen

Conclusions

Finding Equalities:

$Ax \leq b \Rightarrow Ax < b \Rightarrow \text{Conflict} \Rightarrow \text{Equality}$

A Basis for Equalities:

$Ax \leq b \Rightarrow Ax < b \Rightarrow \text{Equality} \Rightarrow \text{Basis}$

Substitution

Applications:

- Simplify for LIA
- Nelson-Oppen for LRA

Thank you for your attention!

Inequality Representation

$$a_i^T \mathbf{x} \leq b_i \qquad \text{for } i = 1, \ldots, m$$

$$a_i^T \mathbf{x} < b_i \qquad \text{for } i = 1, \ldots, m$$

substitution

$$a_i^T \mathbf{x} = b_i \quad \text{for } i \in I \subseteq \{1, \ldots, m\}$$

Tableau & Bounds Representation

$$l_k \leq x_k \leq u_k \quad \text{for } k \in \mathcal{B} \cup \mathcal{N}$$

$$x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad \text{for } i \in \mathcal{B}$$

Equalitiy Test

$$l_k < x_k < u_k \quad \text{for } k \in \mathcal{B} \cup \mathcal{N}$$

$$x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad \text{for } i \in \mathcal{B}$$

Equalitiy Basis

$$x_k = l_k \text{ or } x_k = u_k$$

$$x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad \text{for } i \in \mathcal{B}$$

Inequality Representation

Tableau & Bounds Representation

$$a_i^T \mathbf{x} \leq b_i \qquad \text{for } i = 1, \ldots, m$$

$$l_k \leq x_k \leq u_k \quad \text{for } k \in \mathcal{B} \cup \mathcal{N}$$

$$x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad \text{for } i \in \mathcal{B}$$

Equalitiy Test

$$a_i^T \mathbf{x} < b_i \qquad \text{for } i = 1, \ldots, m$$

$$l_k < x_k < u_k \quad \text{for } k \in \mathcal{B} \cup \mathcal{N}$$

$$x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad \text{for } i \in \mathcal{B}$$

substitution

Equalitiy Basis

pivoting

$$a_i^T \mathbf{x} = b_i \quad \text{for } i \in I \subseteq \{1, \ldots, m\}$$

$$l_k = u_k \qquad \text{for } k \in I \subseteq \mathcal{N}$$

$$x_i = \sum_{j \in \mathcal{N}} a_{ij} \cdot x_j \quad \text{for } i \in \mathcal{B}$$