Reasoning with Sets and Sums of Sets

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Motivation

Ways for Reasoning with Complex Systems

- Introduce calculus/reasoner specifically tailored to the complex system
- Reduce complex system to an established problem (abstraction)

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Boolean Algebra with Presburger Arithmetic

- Introduced by Kuncak et al.
- Transform to equisatisfiable problem in pure arithmetic
- Proven useful for many reasoning task
- Good tool support

Contributions

Boolean Algebra with Presburger Arithmetic and Sums

- Add sum as new bridging function
- Arbitrary mixture of quantified and free set variables

Syntax

$$F ::= A \mid F \wedge F \mid F \vee F \mid \neg F \mid \exists x\ .F \mid \forall x\ .F \mid \exists k\ .F \mid \forall k\ .F$$

$$A ::= B \approx B \mid B \subseteq B \mid$$
$$T \approx T \mid T < T \mid$$
$$B ::= x \mid \emptyset \mid \mathcal{U} \mid B \cup B \mid B \cap B \mid \overline{B}$$
$$T ::= k \mid K \mid \mathrm{MAXC} \mid T + T \mid K \cdot T \mid \mathrm{card}(B)$$
$$K ::= \ldots \mid -2 \mid -1 \mid 0 \mid 1 \mid 2 \mid \ldots$$

Syntax

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Sets are finite.

Extension

BAPA

card with sort: α set $\mapsto \mathbb{N}$, where α: an arbitrary sort

Specific members of the involved sets **do not** matter.

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card with sort: α set $\mapsto \mathbb{N}$, where α: an arbitrary sort

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BAPA_S

sum with sort: β set $\mapsto \beta$, where β: a numerical sort

Specific members of the involved sets **do** matter.

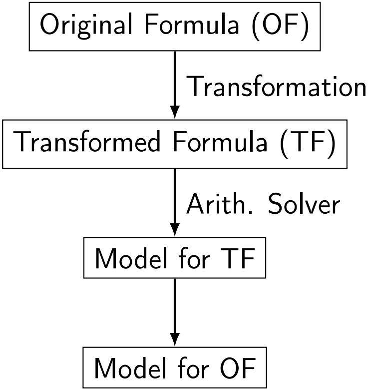
Semantics

Structures

$\mathcal{A} := (\mathbb{N}, \mathbb{F}, \Omega_{\mathcal{A}}, \Pi_{\mathcal{A}})$

- $\mathbb{N}$: only codomain for card
- $\mathbb{F}$: *element support*; elements of sets ($\mathbb{F} \subseteq \mathbb{R}$)
- $\Omega_{\mathcal{A}}$: (more or less) common semantics of functions
- $\Pi_{\mathcal{A}}$: common semantics of predicates

Overview



Transformation:

1. Eliminate $\approx_{\mathsf{set}}$
2. Eliminate $\subseteq$
3. Introduce atomic decompositions
4. Distribute cardinality
5. Purify cardinality
6. **Distribute** sum
7. **Purify** sum
8. **Eliminate quantifiers and add axioms**

Example

Given Formula

$\exists\, x_1 \;\; (x_0 \subseteq x_1 \rightarrow \mathrm{sum}(x_0) \approx \mathrm{sum}(x_0 \cap x_1))$

Example

Eliminate $\approx_{\text{set}}$, and eliminate $\subseteq$

$$\exists\, x_1 \;\; (x_0 \subseteq x_1 \rightarrow \text{sum}(x_0) \approx \text{sum}(x_0 \cap x_1))$$

Example

Eliminate $\approx_{\text{set}}$, and eliminate $\subseteq$

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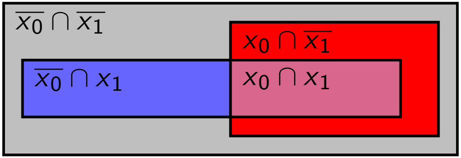
$\exists x_1 \ (x_0 \subseteq x_1 \rightarrow \text{sum}(x_0) \approx \text{sum}(x_0 \cap x_1))$

$\rightsquigarrow$

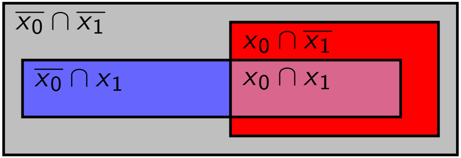
$\exists x_1 \ (\text{card}(x_0 \cap \overline{x_1}) \approx 0 \rightarrow \text{sum}(x_0) \approx \text{sum}(x_0 \cap x_1))$

Atomic Sets and Atomic Decompositions[1]



[1] Ohlbach, Kuncak

Atomic Sets and Atomic Decompositions[1]



$$\mathcal{S}_{00} := \overline{x_0} \cap \overline{x_1}, \quad \mathcal{S}_{01} := \overline{x_0} \cap x_1, \quad \mathcal{S}_{10} := x_0 \cap \overline{x_1}, \quad \mathcal{S}_{11} := x_0 \cap x_1$$

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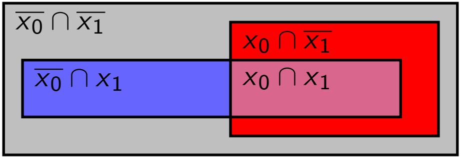

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$$x_0 := \mathcal{S}_{10} \cup \mathcal{S}_{11} = (x_0 \cap \overline{x_1}) \cup (x_0 \cap x_1)$$
$$x_1 := \mathcal{S}_{01} \cup \mathcal{S}_{11} = (\overline{x_0} \cap x_1) \cup (x_0 \cap x_1).$$

[1] Ohlbach, Kuncak

Example

Introduce Atomic Decompositions

$\exists\, x_1 \ \ (\text{card}(x_0 \cap \overline{x_1}) \approx 0 \rightarrow \text{sum}(x_0) \approx \text{sum}(x_0 \cap x_1))$

Example

Introduce Atomic Decompositions

$\exists\, x_1 \;\; (\mathrm{card}(x_0 \cap \overline{x_1}) \approx 0 \rightarrow \mathrm{sum}(x_0) \approx \mathrm{sum}(x_0 \cap x_1))$

$\rightsquigarrow$

$\exists\, x_1 \;\; (\mathrm{card}(\mathcal{S}_{10}) \approx 0 \rightarrow \mathrm{sum}(\mathcal{S}_{10} \cup \mathcal{S}_{11}) \approx \mathrm{sum}(\mathcal{S}_{11}))$

Example

Distribute card, purify card

$\exists\, x_1 \;\; (\mathsf{card}(\mathcal{S}_{10}) \approx 0 \rightarrow \mathsf{sum}(\mathcal{S}_{10} \cup \mathcal{S}_{11}) \approx \mathsf{sum}(\mathcal{S}_{11}))$

Example

Distribute card, purify card

$\exists\, x_1 \;\; (\mathrm{card}(\mathcal{S}_{10}) \approx 0 \rightarrow \mathrm{sum}(\mathcal{S}_{10} \cup \mathcal{S}_{11}) \approx \mathrm{sum}(\mathcal{S}_{11}))$

$\rightsquigarrow$

$\exists\, x_1$

$\exists\, l_{00},\; l_{01},\; l_{10},\; l_{11}$

$\mathrm{card}(\mathcal{S}_{00}) \approx l_{00} \wedge \cdots \wedge \mathrm{card}(\mathcal{S}_{11}) \approx l_{11} \wedge$

$(l_{10} \approx 0 \rightarrow \mathrm{sum}(\mathcal{S}_{10} \cup \mathcal{S}_{11}) \approx \mathrm{sum}(\mathcal{S}_{11}))$

Example

Distribute sum

$\exists\, x_1$

$\exists\, l_{00},\ l_{01},\ l_{10},\ l_{11}$

$\mathsf{card}(\mathcal{S}_{00}) \approx l_{00} \wedge \cdots \wedge \mathsf{card}(\mathcal{S}_{11}) \approx l_{11} \wedge$

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Distribute sum

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$(l_{10} \approx 0 \rightarrow \mathsf{sum}(\mathcal{S}_{10}) + \mathsf{sum}(\mathcal{S}_{11}) \approx \mathsf{sum}(\mathcal{S}_{11}))$

Example

Purify sum

$\exists\, x_1$

$\exists\, l_{00},\ l_{01},\ l_{10},\ l_{11}$

$\mathsf{card}(\mathcal{S}_{00}) \approx l_{00} \wedge \cdots \wedge \mathsf{card}(\mathcal{S}_{11}) \approx l_{11} \wedge$

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Example

Purify sum

$\exists\, x_1$

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Quantifier Elimination[2]

x

[2] Kuncak

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l, k natural numbers
$\text{card}(x) = k + l$.

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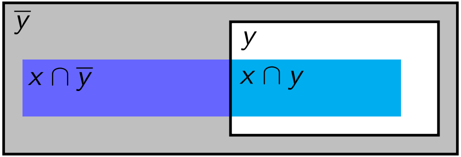

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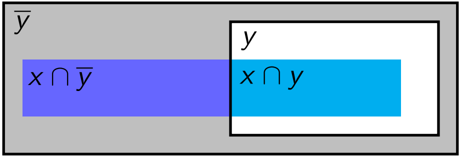

l, k natural numbers
$\text{card}(x) = k + l.$

$\Leftrightarrow$

$\exists y \ (\text{card}(x \cap y) = k \wedge \text{card}(x \cap \overline{y}) = l)$, with y a finite set.

[2]Kuncak

Example

Remove Quantifier (Use Equivalence)

$\exists\, x_1$

$\exists\, l_{00},\ l_{01},\ l_{10},\ l_{11}\ \exists\, s_{00},\ s_{01},\ s_{10},\ s_{11}$

$\mathrm{card}(\mathcal{S}_{00}) \approx l_{00} \wedge \cdots \wedge \mathrm{card}(\mathcal{S}_{11}) \approx l_{11} \wedge$

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$\rightsquigarrow$

$\exists\, l_{00},\ l_{01},\ l_{10},\ l_{11}\ \exists\, s_{00},\ s_{01},\ s_{10},\ s_{11}$

$\mathrm{card}(\overline{x_0}) \approx l_{00} + l_{01} \wedge \mathrm{card}(x_0) \approx l_{10} + l_{11} \wedge$

$\mathrm{sum}(\overline{x_0}) \approx s_{00} + s_{01} \wedge \mathrm{sum}(x_0) \approx s_{10} + s_{11} \wedge$

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$\rightsquigarrow$

$\exists\, l_0,\ l_1,\ \exists\, s_0,\ s_1,$

$\mathsf{card}(\overline{x_0}) \approx l_0 \wedge \mathsf{card}(x_0) \approx l_1 \wedge \mathsf{sum}(\overline{x_0}) \approx s_0 \wedge \mathsf{sum}(x_0) \approx s_1$

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Example

Remove Quantifier (Delete Quantifier)

$\exists\, l_0,\ l_1,\ \exists\, s_0,\ s_1,$

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$\exists\, l_{00},\ l_{01},\ l_{10},\ l_{11}\ \exists\, s_{00},\ s_{01},\ s_{10},\ s_{11}$

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Example

Remove Quantifier (Add Axioms)

$\exists\, l_{00},\ l_{01},\ l_{10},\ l_{11}\ \exists\, s_{00},\ s_{01},\ s_{10},\ s_{11}$

$l_0 \approx l_{00} + l_{01} \wedge l_1 \approx l_{10} + l_{11} \wedge s_0 \approx s_{00} + s_{01} \wedge s_1 \approx s_{10} + s_{11} \wedge$

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$(l_{00} \approx 0 \rightarrow s_{00} \approx 0) \wedge \ldots \wedge (l_{11} \approx 0 \rightarrow s_{11} \approx 0) \wedge$

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$(l_{00} \approx 0 \rightarrow s_{00} \approx 0) \wedge \ldots \wedge (l_{11} \approx 0 \rightarrow s_{11} \approx 0) \wedge$
$(l_{01} \approx 1 \wedge l_{00} \approx 1) \rightarrow (s_{01} \not\approx s_{00}) \wedge \ldots \wedge$
$(l_{11} \approx 1 \wedge l_{10} \approx 1) \rightarrow (s_{11} \not\approx s_{10}) \wedge$
$(l_{10} \approx 0 \rightarrow s_{10} + s_{11} \approx s_{11})$

Example

In Summary

$\exists\, x_1 \;\; (x_0 \subseteq x_1 \rightarrow \text{sum}(x_0) \approx \text{sum}(x_0 \cap x_1))$

$\rightsquigarrow$

$\exists\, l_{00},\; l_{01},\; l_{10},\; l_{11} \;\exists\, s_{00},\; s_{01},\; s_{10},\; s_{11}$

$l_0 \approx l_{00} + l_{01} \wedge l_1 \approx l_{10} + l_{11} \wedge s_0 \approx s_{00} + s_{01} \wedge s_1 \approx s_{10} + s_{11} \wedge$

$(l_{00} \approx 0 \rightarrow s_{00} \approx 0) \wedge \ldots \wedge (l_{11} \approx 0 \rightarrow s_{11} \approx 0) \wedge$

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$(l_{11} \approx 1 \wedge l_{10} \approx 1) \rightarrow (s_{11} \not\approx s_{10}) \wedge$

$(l_{10} \approx 0 \rightarrow s_{10} + s_{11} \approx s_{11})$

(Counter-)Example

Transformation

$(\mathrm{card}(x) \leq 5 \land \mathrm{sum}(x) \leq 5)$

$\rightsquigarrow$

$(l_0 \approx 0 \rightarrow s_0 \approx 0) \land (l_1 \approx 0 \rightarrow s_1 \approx 0) \land$
$(l_0 \approx 1 \land l_1 \approx 1) \rightarrow (s_0 \not\approx s_1) \land$
$(l_1 \leq 5 \land s_1 \leq 5)$

(Counter-)Example

Transformation

$(\mathrm{card}(x) \leq 5 \wedge \mathrm{sum}(x) \leq 5)$

$\rightsquigarrow$

$(l_0 \approx 0 \rightarrow s_0 \approx 0) \wedge (l_1 \approx 0 \rightarrow s_1 \approx 0) \wedge$
$(l_0 \approx 1 \wedge l_1 \approx 1) \rightarrow (s_0 \not\approx s_1) \wedge$
$(l_1 \leq 5 \wedge s_1 \leq 5)$

Elements of sets are natural numbers ($\mathbb{F} = \mathbb{N}$)
$l_0 \mapsto 0 \quad s_0 \mapsto 0 \quad l_1 \mapsto 5 \quad s_1 \mapsto 5$

(Counter-)Example

Transformation

$(\mathrm{card}(x) \leq 5 \wedge \mathrm{sum}(x) \leq 5)$

$\rightsquigarrow$

$(l_0 \approx 0 \rightarrow s_0 \approx 0) \wedge (l_1 \approx 0 \rightarrow s_1 \approx 0) \wedge$
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$l_0 \mapsto 0 \quad s_0 \mapsto 0 \quad l_1 \mapsto 5 \quad s_1 \mapsto 5$

Cannot be extended!

Sum Constructive Structures

Definition (sum constructive)

A structure is called *sum constructive* if and only if its element support $\mathbb{F}$ has the following property:
For all $c \in \mathbb{F}$ there exist infinitely many $a, b \in \mathbb{F}$, such that $a \neq b$ and $a + b = c$.

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Sum constructive : $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \ldots$
Not sum constructive : $\mathbb{N}, \mathbb{R}_+,$ any finite set, $\ldots$

(Counter-)Example

Formula

$$(l_0 \approx 0 \rightarrow s_0 \approx 0) \wedge (l_1 \approx 0 \rightarrow s_1 \approx 0) \wedge$$
$$(l_0 \approx 1 \wedge l_1 \approx 1) \rightarrow (s_0 \not\approx s_1) \wedge$$
$$(l_1 \leq 5 \wedge s_1 \leq 5)$$

Elements of sets are integers ($\mathbb{F} = \mathbb{Z}$)

$l_0 \mapsto 0 \quad s_0 \mapsto 0 \quad l_1 \mapsto 5 \quad s_1 \mapsto 5$

(Counter-)Example

Formula

$$(l_0 \approx 0 \to s_0 \approx 0) \land (l_1 \approx 0 \to s_1 \approx 0) \land$$
$$(l_0 \approx 1 \land l_1 \approx 1) \to (s_0 \not\approx s_1) \land$$
$$(l_1 \leq 5 \land s_1 \leq 5)$$

Elements of sets are integers ($\mathbb{F} = \mathbb{Z}$)
$l_0 \mapsto 0 \quad s_0 \mapsto 0 \quad l_1 \mapsto 5 \quad s_1 \mapsto 5$

Can be extended!

(Counter-)Example

Formula

$(l_0 \approx 0 \to s_0 \approx 0) \land (l_1 \approx 0 \to s_1 \approx 0) \land$
$(l_0 \approx 1 \land l_1 \approx 1) \to (s_0 \not\approx s_1) \land$
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Elements of sets are integers ($\mathbb{F} = \mathbb{Z}$)
$l_0 \mapsto 0 \quad s_0 \mapsto 0 \quad l_1 \mapsto 5 \quad s_1 \mapsto 5$

Can be extended!
$\overline{x} \mapsto \emptyset \quad x \mapsto \{5, 1, -1, 2, -2\}$

Properties

Theorem (Decision Procedure)

If we are considering sum constructive structures and we have a decision procedure for the resulting arithmetical fragment, we have a decision procedure for BAPA_S*.*

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Complexity

Let F be the input formula.

- Size of result is bounded by $O(2^{2\mathrm{size}(F)})$
- Time for transformation bounded by $O(2^{2\mathrm{size}(F)})$
- Number of quantifier alternations does not change

Conclusion

- Introduced BAPA_S as extension of BAPA
- Presented a transformation from BAPA_S to pure arithmetic
- Decision procedure for BAPA_S[3]
- Properties are proven
- Implementation is ongoing

[3] Under certain conditions

Outlook

- min and max as additional bridging function
- Developement of a general approach for new bridging functions
- Consider intervals instead of sets and try to extend the developed methods
- Consider if the developed methods can be extend for reasoning with the *duration calculus*

The constant $\mathcal{U}$

- $\mathcal{U}_{\mathcal{A}} \in \mathcal{P}_f(\mathbb{F})$,
- $\mathrm{MAXC} := \mathrm{card}(\mathcal{U})$
- $\mathrm{MAXS} := \mathrm{sum}(\mathcal{U})$
- $\mathrm{MAXC}_{\mathcal{A}} = \mathrm{card}_{\mathcal{A}}(\mathcal{U}_{\mathcal{A}}) := |\mathcal{U}_{\mathcal{A}}|$
- $\mathrm{MAXS}_{\mathcal{A}} = \mathrm{sum}_{\mathcal{A}}(\mathcal{U}_{\mathcal{A}}) := \sum_{e \in \mathcal{U}_{\mathcal{A}}} e$
- $\forall x \; p(x)$ true iff $p_{\mathcal{A}}(o)$ true for all $o \in \mathcal{P}(\mathcal{U}_{\mathcal{A}})$
- $\exists x \; p(x)$ true iff $p_{\mathcal{A}}(o)$ true for at least one $o \in \mathcal{P}(\mathcal{U}_{\mathcal{A}})$
- $o \in \mathcal{P}_f(\mathbb{F})$, then $\complement_{\mathcal{A}} o := \{e \mid e \in \mathcal{U}_{\mathcal{A}} \text{ and } e \notin o\}$.

Duration Calculus

- Temporal logic
- Zhou Chaochen, C. A. R. Hoare, and Anders P. Ravn (1991)
- Talks about intervals
- Example: $(l \geq 60) \rightarrow (\int \text{Leak} \leq 0.05 * l)$