Complete Instantiation of Quantified Formulas in Satisfiability Modulo Theories

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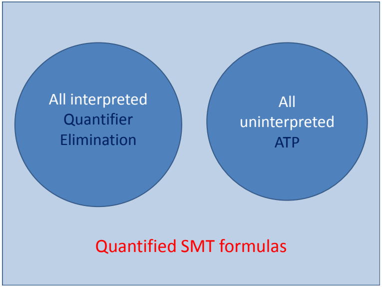
- Many more......
  - Heaps, linked lists,...

Quantifier Reasoning in SMT: a long-standing challenge

- Mixed uninterpreted and interpreted symbols



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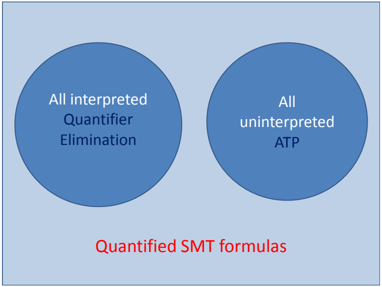

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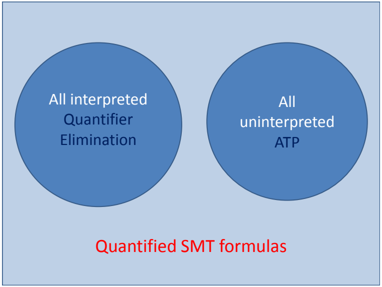

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- Solutions
  - Theory resolution, SMT+ATP,...

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Example

$f(a) < 1 \wedge (\forall x : f(x) > 2)$

- Select a as the ground term for instantiation
- Instantiate $\forall x : f(x) > 2$ with x substituted by a
- $f(a) < 1 \wedge f(a) > 2$, contradiction

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What if P' is satisfiable?

Incomplete vs Complete Instantiation

- Instantiation based methods are attractive
- Acceptable performance (E-matching,...)
- Problem:Incompleteness
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In this talk, we will introduce a series of new fragments that can be solved by complete instantiation

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Can we have a complete method based on instantiation?

Ideally, given F, we would like to have a F^* such that :

- F^* is quantifier free
- F^* is the conjunction of instantiations of F
- F and F^* are equi-satisfiable

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Of course, only possible for formulas in some fragments

- Array property fragment by Bradley et al
- Linked list by Scott et al

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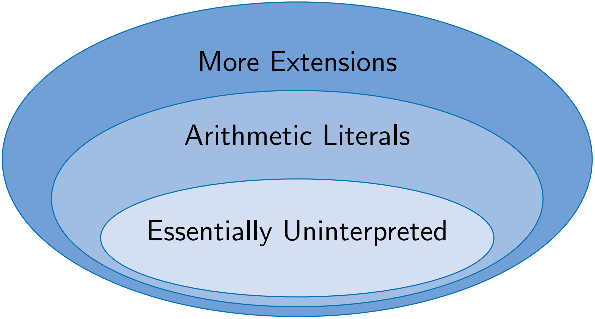
New Fragments for Complete Instantiation



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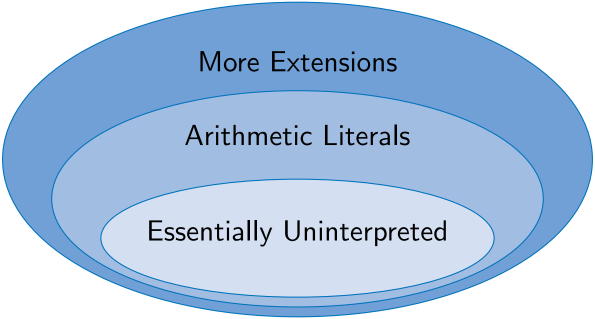

Two key issues for complete instantiation:

- What to instantiate? How to collect ground terms for instantiation?
- Why complete? How to prove?

Notations

- $x, y, x_1, y_1, \ldots$ denotes variables
- $a, b, c, \ldots$ are constants
- $f, g, h, \ldots$ are uninterpreted functions
- $+, -, *, <, \leq, \ldots$ are interpreted arithmetic symbols
- t^M denotes the interpretation of term t in model M

Essentially Uninterpreted Formulas

Definition (Essentially Uninterpreted)

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Example

- $f(x) + b > c$, YES
- $f(x + y) > c$, NO
- A formula in pure first order logic is an EU formula

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Is $P(f(b)) \wedge Q(f(a)) \wedge \forall x : P(f(x))$ satisfiable?

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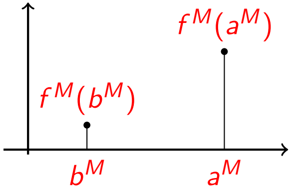

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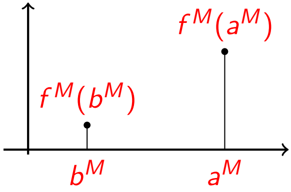

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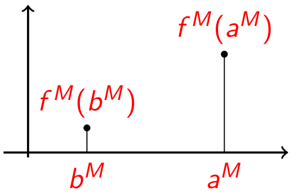

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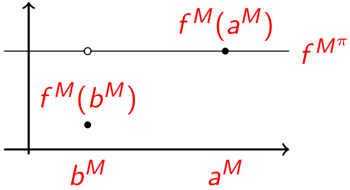

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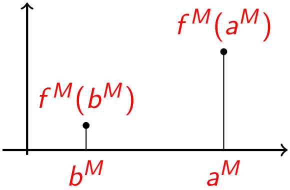

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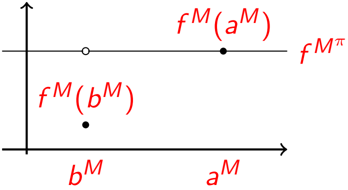

- We have $P^{M^\pi}(f^{M^\pi}(e))$ holds for all e in the domain

More notations

From now on, assume:

- A formula is a set of CNF clauses
- A clause is universally quantified at outermost scope
- $t[x_1, x_2, ..., x_n]$ means term t may contains variables $x_1, x_2, ..., x_n$
- $t[x/t_1]$ is the result of substituting t_1 for all free occurrences of x
- $t[x_1/s_1, x_2/s_s, ..., x_n/s_n]$ with the obvious meaning
- $t[S_1, S_2, ..., S_n]$ denotes the set $\{t[x_1/s_1, x_2/s_s, ..., x_n/s_n] \mid s_i \in S_i\}$, where S_i are sets of terms

Rules for Collecting Ground Terms for EU Formulas

- S_i and A_f are sets of ground terms
- Details skipped
- S_i contains ground terms for instantiating variable x_i
- A_f contains all ground terms that can appear as argument of f in the result of instantiation

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- S_i contains ground terms for instantiating variable x_i
- A_f contains all ground terms that can appear as argument of f in the result of instantiation
- Suppose $f(t)$ appears in the quantified formula :

t is a ground term	A_f includes t
t is $s[x_1, ..., x_n]$	A_f contains $s[S_1, ..., S_n]$
t is variable x_j	A_f equals to S_j

EU Example

F

$$g(x_1) \leq 0$$
$$g(f(x_2)) + 1 \leq f(x_2)$$
$$f(a) = 0$$

Constraints —— Rules used

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$A_g = S_1$	x_1 is a variable
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- The ground terms are collected incrementally

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M

Let $a = 2$
$f(a) = 0$
$g(0) = -1$

EU Example (Cont.)

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M^π

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Proof of Completeness

- Details skipped
- The big idea:
  - Construct M^π from M by defining interpretation for uninterpreted functions (projection)
  - Assume F^* is satisfiable but F is not
  - Deduce contradiction

Theorem

Given F an essentially uninterpreted formula, F and F^* are equi-satisfiable

Finite F^*

- If F^* is finite, then we have a decision procedure
- When F^* is finite?
  - The set $\{S_i\}$ is stratified
  - Details skipped
  - Better than sorts stratification
    - $f(a) = a \wedge g(f(x)) = f(x)$
    - a and $f(a)$ must be in the same sort

Herbrand Theorem and Finite F^*

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- Do we have a new decidable class (stratified) in pure first order logic?
- Yes, we have

Refinement: Lazy construction of F^*

- F^* may be very big (even infinite)
- By following the rules for collecting ground terms, incrementally construct sequence $F^0 \subset F^1 \subset \ldots$
- If F^k is unsatisfiable, then return unsatisfiable
- If F^k is satisfiable, construct candidate model M^{π^k}
  - If M^{π^k} is a model for all quantified formulas, return satisfiable
  - If M^{π^k} is not a model for all quantified formulas, continue

Refinement: Model Checking Example

How to check if M is a model of a quantified formula $\forall P$?

- Model M
  $h = \lambda x.$ IF $(x = 2)$ THEN 0 ELSE 1
  $g = \lambda x, y.$ IF $(x = 0$ AND $y = 2)$ THEN -1 ELSE 0
- Quantified Formula
  $\forall x1, x2 : g(x1, x2) = 0 \vee h(x2) = 0$

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- Check if valid (Send it to a SMT solver)
- The above formula is valid and we conclude that M is indeed a model for the quantified formula

Refinement: Model Based Instantiation

- Model checking can be used to select ground terms for instantiation

Example

F	$f(x) \leq 0,\ f(a) = 1, f(b) = -1$
F^0	$f(a) = 1, f(b) = -1$
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F^1	$f(a) = 1, f(b) = -1,\ f(a) < 0$

- Unsatisfiable

Infinite F^* and Refutation Complete

When F^* is infinite, is the procedure refutation complete?

F : unsatisfiable, x ranges over integers

$f(x_1) < f(f(x_1))$	f is always increasing
$f(x_2) < a$	f has a up limit
$1 < f(0)$	f has a bottom

F^* : every finite subset is satisfiable

$f(0) < f(f(0)), f(f(0)) < f(f(f(0))), \ldots$

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- Refutation complete, if we assume the background theory is a (potentially infinite) set of sentences
- Refutation incomplete, if the background theory is a class of structures (Why? Compactness no longer holds)

Arithmetic Literals

Definition (Arithmetic Literals)

Variables in a CNF clause may appear in literals of the form:
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Example

$\neg(x_i \leq x_j) \vee A[x_i] \leq A[x_j]$

More Extensions

- Offsets
  - $x_i + 2$
  - $\neg(0 \leq x_i) \vee \neg(x_i \leq n) \vee f(x_i) = g(x_i + 2)$
- Modular equalities
  - $\neg(x_i =_n t)$, means $x_i = t + n * c$, n is an integer and c is a constant
  - $\neg(x_1 =_4 0) \vee (star(x_1) = e)$
- Pseudo-macros

Related Works

- Array property fragment
  - No nested application of uninterpreted functions
  - $P(f(g(x)))$, NO
- "What else is decidable" by Habermehl et al
  - Literals of the form $x_i - x_j \leq c$ are allowed
  - In a clause, at most one literal of the form $f(x_i) - g(x_j) \leq c$ is allowed
  - No other type literal allowed, no nested applications
  - Proof based on a customized automaton, implementation unknown
- Local Theories
  - Certain quantified formulas can be added upon other decidable fragments

Summary

- Several new fragments that can be decided by complete instantiation
- Model checking and model based instantiation
- Conditions for refutation complete
- Z3 was the only solver could return SAT for satisfiable quantified formulas in SMT COMP 2008

Future works:

- Efficient implementation
- More fragments for complete instantiation

Thank you

- Questions?